Tensor notation

Covariant and Contravariant vectors

Change of basis

Say we have a vector space V (in field F) described by the basis $B_{old} = (v_1, \dots, v_n)$

We can express new basis vectors $B_{new}(w_1, \dots, w_n)$ wrt the old basis as follows: $\hat{\omega}_j = \sum_i a_{ij}\hat{v}_i$

↳ $a_{i,j}$ are the coordinates of the j-th new basis vector w_j wrt the i-th old basis vector v_i

A vector $\vec{z}$ in V can then be described by: $\vec{z} = \sum_i x_i \hat{v}_i = \sum_j y_j \hat{\omega}_j$

As $\hat{\omega}_j = \sum a_{i,j}\hat{v}_i$ we have $x_i = \sum_j a_{ij} y_j$

That is: $\vec{z}_{new} = A\,\vec{z}_{old}$ or $\vec{x} = A\vec{y}$

Consequence

Basis transforms as follows: $W = A^T V$ where $W = [\hat{\omega}_1, \hat{\omega}_2, \dots, \hat{\omega}_n]^T$ and $V = [\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n]^T$

e.g. $\hat{\omega}_1 = a_{11}\hat{v}_1 + a_{21}\hat{v}_2$ and $\hat{\omega}_2 = a_{12}\hat{v}_1 + a_{22}\hat{v}_2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix} = A^T V = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix}$$

On the other hand a vector changes as $\vec{y} = A^{-1}\vec{x}$ where y_i is the i-th coordinate in new basis while x are in old basis

That is, a vector transforms in the opposite way w.r.t the basis vectors i.e. Contravariant

Tensors

A tensor is denoted by a symbol and collection sub-/super-scripts

Types:

- Tensor of rank-0 ⟹ scalar e.g. ϕ
- Tensor of rank-1 ⟹ vector e.g. x_μ, x^μ
- Tensor of rank-2 ⟹ tensor e.g. σ_{ij}, σ^{ij}

All vectors in euclidean space are contravariant due to the metric being diag(1, 1, 1)

Vectors

Types of vectors:

↳ Contravariant $x^\mu, A^\mu, \dots$ i.e. Column Vectors

↳ Transforms opposite to basis vectors

↳ $(x')^\nu = \Lambda^\nu_{\ \mu} x^\mu$ or $(x')^\nu = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right) x^\mu$ ⟶ This relation is valid for every contravariant vector

↳ Covariant $x_\mu, A_\mu, \dots$

↳ Transforms like basis

↳ $(x'_\nu) = (\Lambda^{-1})^\nu_{\ \mu} x_\mu$ or $(x'_\nu) = \left(\frac{\partial x_\mu}{\partial x'_\nu}\right) x_\mu$ ⟶ This relation is valid for every covariant vector

Similarly:

$T'^{\mu\nu} = \Lambda^\mu_{\ \alpha}\Lambda^\nu_{\ \beta} T^{\alpha\beta}$

Dot product: $x^\mu \cdot x^\nu = x_\nu x^\nu = \eta_{\mu\nu} x^\mu x^\nu$

Matrix product: $\omega_i = \sigma_{ij} u_j$ in euclidean space

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \implies \vec{\nabla}\delta_{ij} = \frac{\partial \delta_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} = \vec{\nabla} \quad \text{and} \quad \eta_{\mu\nu}\eta^{\mu\nu} = \eta^\mu_{\ \nu} = \delta^\mu_{\ \nu}$$

$\partial_\mu x^\nu = \delta^\nu_\mu$ and $\partial_\mu x_\nu = \eta_{\mu\nu}$

$\frac{\partial(\partial_\alpha x_\beta)}{\partial(\partial_\mu x_\nu)} = \delta^\mu_\alpha \delta^\nu_\beta$

Levi-Civita

$\vec{c} = \vec{a} \times \vec{b}$ i.e. $c_i = \varepsilon_{ijk} a_j b_k$

$$\varepsilon_{ijk} \begin{cases} 0 & \text{if repeated index} \\ +1 & \text{if even permutation} \\ -1 & \text{if odd permutation} \end{cases}$$

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$

$\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1$

e.g. $\varepsilon_{112} = 0$

Lecture 06.02.2024

Limits and Scales

Natural Units: $c = \hbar = k_B = 1$

$[c] = L T^{-1}$

$[\hbar] =$

$[G] =$

$[\text{Energy}] = [\text{Mass}] = [\text{Temperature}] = [\text{Length}]^{-1} = [\text{Time}]^{-1}$

Compton Wavelength: $\lambda_c = \dfrac{2\pi\hbar}{mc}$

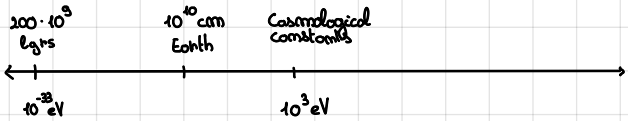
$G = (\hbar c) M_P^{-2} = M_P^{-2}$ $\quad M_P$ Planck Mass

Planck Scale:

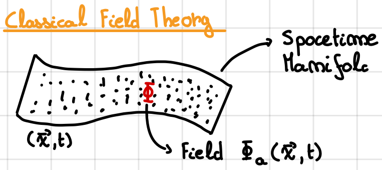
$M_P \approx 10^{19}$ GeV

$l_P \approx 10^{-33}$ cm

$t_P \approx 10^{-44}$ s



Classical Field Theory



The field can be e.g. $\vec{E}(x,t)$, $\vec{B}(x,t)$

We can merge this in a 4-Vector $A^\mu(\vec{x}, t) = (\phi, \vec{A})$

$\phi \rightarrow 0^{th}$ Component

$\vec{A} \rightarrow$ 3 Vector

Maxwell's equations

$\vec{E} = -\vec{\nabla}\phi - \dfrac{\partial \vec{A}}{\partial t}$ $\qquad \vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{\nabla} \cdot \vec{B} = 0$ $\qquad \dfrac{d\vec{B}}{dt} = -\vec{\nabla} \times \vec{E}$

Lagrangian

Lagrangian: $L(t) = \int \mathcal{L}(\phi_a, \partial_\mu \phi_a)\, dx^3$

Lagrangian Density: $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$

Action: $S = \int_{t_1}^{t_2} dt \int d^3x\, \mathcal{L}($

$S_d = \int d^d \mathcal{L}$

The Lagrangian Dens. depends on an arbitrary 4-vec field, on its time derivative but also on its gradient instead of depending on $q, \dot{q}$ as in classical dynamics. Why is that the case? Unlike discrete mechanics, fields contain large numbers of particles ($\sim$ continuous medium). As such, some properties will be described by a gradient

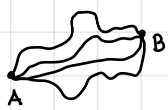
In field theory: $\mathcal{L}(\vec{\nabla}\phi, \nabla^2\phi, \nabla^3\phi, \dots)$ instead of $\mathcal{L}(q, \dot{q})$

- Higher derivatives bring issues such as "Ghosts" which are unphysical states
- While in prin. we can deal with infinite derivatives, we tend to not consider infinite time derivatives as they make $\hat{H}$ unbounded from below i.e. no bound state

Note: Lagrangian and Action must be invariant under the Lorentz group operations (i.e. Lorentz Invariant)

2 Aspects of a system

- Kinematics:
- Dynamics: How system evolves over time



Principle of least action: A system will evolve according to the path that minimizes the action

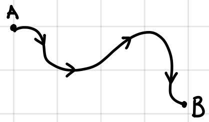
i.e. $\delta S = 0$ when going from A to B

Application of Principle of Least Action

Action: $S = \int d^4x\, \mathcal{L}$ where $\mathcal{L}$ is the Lagrangian Density

By principle of least action:

$$\delta S = \int d^4x\, \delta\mathcal{L} = 0 \qquad \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)$$



Exploiting Multiplication rule we have:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi$$

Then, we can rewrite the action as follows:

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \int d^4x\, \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] =$$

As all paths have fixed endpoints (i.e. A, B) at those endpoints $\delta\phi = 0$

$$= \int d^4x \left\{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \underbrace{\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right]}\right\} =$$

$$= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi = 0$$

Changes of the Lagrangian by a total derivative do not affect δS if $\delta\phi_A = \delta\phi_B = 0$ which is the necessary condition for the new path to correspond to the old path at the end points

In order for $\delta S = 0$ for all paths with fixed endpoints A, B and $\delta\phi(A) = \delta\phi(B) = 0$, we need: $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$ "Euler-Lagrange Equation"

Example: Klein-Gordon Equation

Lagrangian of a real scalar field: $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2$

Apply Eul-Lag. Eq. wrt $\phi(\vec{x}, t)$:

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} = \frac{1}{2}\eta^{\mu\nu}\left[\frac{\partial(\partial_\mu\phi)}{\partial(\partial_\alpha\phi)}\partial_\nu\phi + \partial_\mu\phi\frac{\partial(\partial_\nu\phi)}{\partial(\partial_\alpha\phi)}\right] = \frac{1}{2}\eta^{\mu\nu}\left[\delta^\mu_\alpha\,\partial_\nu\phi + \partial_\mu\phi\,\delta^\nu_\alpha\right]$$

$$\partial_\alpha\left[\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\right] = \frac{1}{2}\eta^{\mu\nu}\left[\delta^\mu_\alpha\partial_\alpha\partial_\nu\phi + \delta^\nu_\alpha\partial_\alpha\partial_\mu\phi\right] = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = \partial^\mu\partial_\nu\phi = \Box\phi$$

$$\text{E.O.M}: \quad \Box\phi + m^2\phi = \ddot{\phi} - \nabla^2\phi + m^2\phi = 0$$

For $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)$ we get $\Box\phi + \frac{\partial V}{\partial\phi} = 0$

Example: First order Lagrangian

Consider the Lagrangian: $\mathcal{L} = \frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - \vec{\nabla}\psi^*\cdot\vec{\nabla}\psi - m\psi^*\psi$

We need to treat ψ^* and ψ separately as they have different dependencies due to complex conjugacy:

$$\partial_\mu\psi = \dot{\psi} + \vec{\nabla}\psi$$
$$\partial_\mu\psi^* = \dot{\psi}^* + \vec{\nabla}\psi^*$$

$$\frac{\partial\mathcal{L}}{\partial\psi} = -\frac{i}{2}\dot{\psi}^* - m\psi^* \qquad \frac{\partial\mathcal{L}}{\partial\psi^*} = \frac{i}{2}\dot{\psi} - m\psi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}, \frac{\partial\mathcal{L}}{\partial\vec{\nabla}\psi}\right) = \left(\frac{i}{2}\psi^*, -\vec{\nabla}\psi^*\right) \Longrightarrow \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \frac{i}{2}\dot{\psi}^* - \nabla^2\psi^*$$

Because of same index we get $\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} + \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\psi)}$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)} = \left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}^*}, \frac{\partial\mathcal{L}}{\partial\vec{\nabla}\psi^*}\right) = \left(-\frac{i}{2}\psi, -\vec{\nabla}\psi\right) \Longrightarrow \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)} = \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\psi^*)} + \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\psi^*)} = -\frac{i}{2}\dot{\psi} - \nabla^2\psi$$

E.O.M for ψ: $i\dot{\psi}^* + m\psi^* - \nabla^2\psi^* = 0$

E.O.M for ψ^*: $-i\dot{\psi} + m\psi - \nabla^2\psi = 0$

Example: Maxwell's equations

Proca Lagrangian: $\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2$

Notice that:

$$\partial_\mu A_\nu = (\partial_0 A_0 + \partial_i A_0, \partial_0 A_i + \partial_i A_i)$$
$$\partial^\mu A^\nu = (\partial^0 A^0 + \partial^i A^0, \partial^0 A^i + \partial^i A^i) = (\partial_0 A^0 - \partial_i A^0, \partial_0 A^i - \partial_i A^i)$$

$$(\partial_\mu A^\nu) = (\partial_0 A_0)(\partial_0 A^0) - (\partial_0 A_0)(\partial_i A^0) + (\partial_i A_0)(\partial_0 A^0) - (\partial_i A_0)(\partial_i A^0) +$$
$$+ (\partial_0 A_i)(\partial_0 A^i) - (\partial_0 A_i)(\partial_i A^i) + (\partial_i A_i)(\partial_0 A^i) - (\partial_i A_i)(\partial_i A^i)$$
$$= \eta^{00}\dot{A}_0^2 - \eta^{00}\dot{A}_0\vec{\nabla}A_0 + \eta^{00}\dot{A}_0\vec{\nabla}A_0 - \eta^{00}\nabla^2 A_0 + \eta^{ii}\dot{A}_i^2 - \eta^{ii}\dot{A}_i\vec{\nabla}A_i + \eta^{ii}\dot{A}_i\vec{\nabla}A_i - \eta^{ii}\nabla^2 A_i \qquad \text{As } A^\mu = \eta^{\mu\nu}A_\nu$$
$$= \dot{A}_0^2 - \dot{A}_i^2 - \nabla^2 A_0 + \nabla^2 A_i$$
$$(\partial_\mu A^\mu)^2 = \dot{A}_0^2 - \nabla^2 A_i$$

The Lagrangian becomes: $\mathcal{L} = \frac{1}{2}\dot{A}_i^2 - \nabla^2 A_i + \frac{1}{2}\nabla^2 A_0$

Let's write the Lagrangian in a more useful way:

$\hookrightarrow \partial^\mu A^\nu = \eta^{\mu\alpha}\eta^{\nu\beta}\partial_\alpha A_\beta$

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\mu A_\nu)(\partial_\alpha A_\beta) + \frac{1}{2}(\eta^{\mu\beta}\partial_\mu A_\beta)^2$$

Let's exploit Euler Lagrange:

$$\frac{\partial \mathcal{L}}{\partial A^\gamma} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\omega A_\gamma)} = -\frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\left[\frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\omega A_\gamma)}(\partial_\alpha \partial A_\beta) + (\partial_\mu A_\nu)\frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\omega A_\gamma)}\right] + (\eta^{\mu\beta}\partial_\mu A_\beta)\frac{\partial(\eta^{\mu\beta}\partial_\mu A_\beta)}{\partial(\partial_\omega A_\gamma)} =$$

$$= -\frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\left[\delta^\omega_\mu \delta^\nu_\gamma (\partial_\alpha \partial A_\beta) + (\partial_\mu A_\nu)\delta^\alpha_\omega \delta^\beta_\gamma\right] + (\partial_\mu A^\mu)\eta^{\mu\beta}\delta^\mu_\omega \delta^\beta_\gamma =$$

$$= -\frac{1}{2}\left[\eta^{\omega\alpha}\eta^{\gamma\beta}(\partial_\alpha A_\beta) + \eta^{\mu\omega}\eta^{\nu\gamma}(\partial_\mu A_\nu)\right] + (\partial_\mu A^\mu)\eta^{\omega\gamma}$$

$$= -(\partial^\omega A^\gamma) + (\partial_\mu A^\mu)\eta^{\omega\gamma}$$

$$\partial_\omega\left(\frac{\partial \mathcal{L}}{\partial(\partial_\omega A_\gamma)}\right) = -\partial_\omega \partial^\omega A^\gamma + \partial_\omega \partial_\mu A^\mu \eta^{\omega\gamma} =$$
$$= -\partial_\omega \partial^\omega A^\gamma + \partial^\gamma \partial_\mu A^\mu =$$
$$= \partial_\mu(\partial^\gamma A^\mu) - \partial_\omega(\partial^\omega A^\gamma) =$$
$$= \partial_\mu(\partial^\gamma A^\mu - \partial^\mu A^\gamma) = -\partial_\mu F^{\mu\nu} \qquad \omega \longmapsto \mu \text{ as all } \omega \text{ are contracted}$$

Field Strength Tensor:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{when } F^{i0} = E^i \ (i \in \{1,2,3\}) \text{ and } F^{ij} = -\varepsilon^{ijk}B^k \qquad F^{\mu\mu} = 0$$

$$F^{\mu\nu}F_{\mu\nu} = (\partial^\mu A^\nu)(\partial_\mu A_\nu) - (\partial^\mu A^\nu)(\partial_\nu A_\mu) - (\partial^\nu A^\mu)(\partial_\mu A_\nu) + (\partial^\nu A^\mu)(\partial_\nu A_\mu) =$$
$$= (\partial_\mu A_\nu)^2 + (\partial_\nu A_\mu)^2 - 2(\partial^\mu A^\nu)(\partial_\nu A_\mu) =$$
$$= 2\left[(\right.$$

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \Longrightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 2F^{\mu\nu} - 2F^{\nu\mu} = 4F^{\mu\nu} \quad \text{as } F^{\mu\nu} = -F^{\nu\mu}$$

Lecture 8.02.2024

Lagrangian

Once you have your Lagrangian, you can derive the equations of motion

↳ The E.-L. equations will be the same as a point particle but there will also be a term depending on the spatial gradient due to the presence of a field

↳ While signs of the Lagrangian might on the metric signature, equations of motion will not

The Lagrangian is defined as $L = T - U$

e.g. $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}m^2\phi^2 = \pm\frac{1}{2}\dot{\phi}^2 \mp (\vec{\nabla}\phi)^2 - \frac{1}{2}m\phi^2$ $\qquad \Box = \partial^\nu\partial_\nu = \eta^{\mu\nu}\partial_\mu\partial_\nu = \eta^{00}\partial_0^2 + \eta^{ii}\partial_i^2 = \frac{\partial^2}{\partial t^2} - \nabla^2$

$T = \int d^3x\, \frac{1}{2}\dot{\phi}^2$

$U = \int d^3x \left[\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$

Applying E-L equations: $\ddot{\phi} - \nabla^2\phi = -m^2\phi$ i.e. $\Box\phi + m^2\phi = 0$

In order to apply quantisation of the field we need to move from real space to momentum space

↳ Change of basis through Fourier Transform applied to equations of motion

For a generic Lagrangian: $\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\phi - V(\phi)$ we get $\partial_\mu\partial^\mu\phi + \frac{\partial V}{\partial\phi} = 0$

Harmonic Oscillators and fields: Check

Complex Scalar Field

$\mathcal{L} = \frac{i}{2}(\varphi^*\dot{\varphi} - \dot{\varphi}^*\varphi) - \vec{\nabla}\varphi^*\cdot\vec{\nabla}\varphi - m\varphi^*\varphi$

$\mathcal{L} \propto (\varphi, \varphi^*, \partial_\mu\varphi, \partial_\mu\varphi^*)$

Maxwell Lagrangian

$A^\mu = (\phi, \vec{A})$

Field Strength: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

The field strength and A^μ are Lorentz Invariant but ϕ and $\vec{A}$ are not.

$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2$ "Proca Lagrangian"

$\mathcal{L} \sim \frac{1}{2}\dot{A}_i^2$ but no $\dot{A}_0^2$ term i.e. no kinetic term in ϕ

In addition, $\mathcal{L}$ has no term proportional to $A^2 = A_\mu A^\mu$ i.e. no mass term $\Longrightarrow$ Field quant has 0 mass

$(\partial_\mu A^\nu)(\partial_\mu A^\nu) =$

Maxwell's equation of motion: $\partial_\mu F^{\mu\nu} = 0$ and $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

Locality

Lagrangian is local

↳ Non-Locality: Events can influence other events immediately even though they are very far away i.e. Every event is causally connected to previous events

Locality: Only events within light cone are causally connected

↳ There are no terms connecting two arbitrary positions e.g. no terms like $L = \int d^3x\, d^3y\, \phi(x)\phi(y)$

↳ Closest non-locality given by gradient which connects $\vec{x}$ to $\vec{x} + \delta\vec{x}$

Lorentz Invariance

- Laws of nature are relativistic i.e. independent of inertial reference frame
- Lorentz transformations include:

1) Boosts

e.g. $\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ i.e. Boost along x-axis

2) Rotations

e.g. $\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ i.e. rotation about z axis

$$ds^2 = dx^{\mu}dx_{\nu} = \eta^{\mu\nu}dx_{\mu}dx_{\nu} = dt^2 - (d\vec{x})^2$$

$$\partial_{\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\Box = \partial^{\mu}\partial_{\mu} = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

$$\Lambda^{\mu}{}_{\sigma}\,\eta^{\sigma\tau}\,\Lambda^{\nu}{}_{\tau} = \eta^{\mu\nu}$$

Lorentz transf: $x^{\mu} \longmapsto \Lambda^{\mu}{}_{\nu}x^{\nu}$

$$\phi(x) \longmapsto \underbrace{\phi'(x)}_{\text{Shifted field at old position}} = \overbrace{\phi(\Lambda^{-1}x)}^{\text{old field at the equivalent location in unshifted frame}}$$

Active transformation

Active transf: Transform field

Passive transf: Transform reference frame

Vector fields transform covariantly or contravariant wrt basis vectors:

Contravariant: $A^{\mu}(x) \longmapsto \Lambda^{\mu}{}_{\nu}A^{\nu}(\Lambda^{-1}x)$

Covariant : $A_{\mu}(x) \longmapsto (\Lambda^{-1})^{\mu}{}_{\nu}A_{\nu}(\Lambda^{-1}x)$

Example: Klein-Gordon Equation is Lorentz Invariant

Applying Lorentz Transform to Scalar Field: $\phi(x^{\mu}) \longrightarrow \phi'(x^{\mu}) = \phi\left((\Lambda^{\nu}{}_{\mu})^{-1}x^{\mu}\right)$ N.B. x^{μ} is contravariant

↳ For the sake of simplicity we will write from now on: $\phi(x) \longmapsto \phi'(x) = \phi(y)$ where $y = \Lambda^{-1}x$

What about the derivative $\partial_{\mu}\phi$? $\partial_{\mu}\phi$ is a covariant quantity so it should transform just like basis vectors

That is: $\partial_{\nu}\phi'(x) = \Lambda^{\nu}{}_{\mu}\partial_{\mu}\phi(x)$ or $\partial_{\mu}\phi(x) \longmapsto (\Lambda^{-1})^{\nu}{}_{\mu}(\partial_{\nu}\phi'(x)) = (\Lambda^{-1})^{\nu}{}_{\mu}(\partial_{\nu}\phi(y))$

Looking at the Klein-Gordon Lagrangian $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}m^2\phi^2$ we get:

$$\begin{aligned} \mathcal{L}(x) \longmapsto \mathcal{L}(y) &= \frac{1}{2}\eta^{\mu\nu}(\Lambda^{-1})^{\alpha}{}_{\mu}(\Lambda^{-1})^{\beta}{}_{\nu}(\partial_{\alpha}\phi)(\partial_{\beta}\phi) - \frac{1}{2}m^2\phi^2(y) = \\ &= \frac{1}{2}(\Lambda^{-1})^{\alpha}{}_{\mu}\eta_{\mu\nu}(\Lambda^{-1})^{\beta}{}_{\nu}(\partial_{\alpha}\phi)(\partial_{\beta}\phi) - \frac{1}{2}m^2\phi^2(y) = \\ &= \frac{1}{2}\eta^{\alpha\beta}(\partial_{\alpha}\phi(y))(\partial_{\beta}\phi(\beta)) - \frac{1}{2}m^2\phi^2(y) = \mathcal{L}(y) \end{aligned}$$

As Lagrangian is Lorentz invariant, to have action be inv. we need $d^4x = d^4y$ where $y = \Lambda^{-1}x$ i.e. Jacobian $J = 1$

The Jacobian will not be exactly 1 but correction is very tiny $\Longrightarrow$ Lorentz variant corrections are small and can be invariant

e.g. $y = x + \delta x \Longrightarrow \frac{\partial y^{\mu}}{\partial x^{\mu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\delta x^{\mu}$ and $J = \det\left(\frac{\partial y^{\mu}}{\partial x^{\nu}}\right) = 1 + \partial_{\nu}(\delta x^{\mu}) \approx 1$

N.B. Not all Lagrangians are Lorentz invariant. For a Lorentz invariant Lagrangian we need for time and space to be on equal footing i.e. All indices should be contracted by means of Lorentz inv. object such as η. If Lagrangian is invariant then so is the action for reasons discussed above

e.g. First order Lagrangian is not Lor. inv. as it is linear in time derivatives while it is quadratic in spatial der. (i.e. No proper contraction)

e.g. Maxwell Lagrangian is Lorentz inv. as all indices are contracted. Check by doing $A^{\mu}(x) \longrightarrow \Lambda^{\mu}{}_{\nu}A^{\nu}(\Lambda^{-1}x)$

Lecture 13.02.2024

Noether's Theorem

Relates symmetries of Action (i.e. Lorentz symmetry, internal symmetries, Gauge symmetries, ...) to conserved quantities

↳ For every continuous symmetry of the Lagrangian there exists a conserved current J^μ such that $\partial_\mu J^\mu = \frac{\partial J^0}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

↳ Action?

N.B. Conserved currents imply conserved charge Q, but conservation of current is a stronger statement as it implies that charge is conserved locally

$$Q = \int d^3x\, j^0 \implies \frac{dQ}{dt} = \int d^3x \frac{dj^0}{dt} = -\int d^3x\, \vec{\nabla}\cdot\vec{j} = 0 \quad \text{assuming } \vec{j} \longrightarrow 0 \text{ as } |\vec{x}| \longrightarrow \infty$$

In a volume V:

$$Q_V = \int_V d^3x\, j^0 \implies \frac{dQ_V}{dt} = -\int_V d^3x\, \vec{\nabla}\cdot\vec{J} = -\int_A \vec{J}\cdot d\vec{S}$$

Any charge leaving V must be accounted for by a 3-vector current $\vec{J}$ out of A

↳ Local Charge

Proof:

Let's consider a transformation of the following type:

$x^\mu \longmapsto x^\mu + \delta x^\mu$ and $\phi_a \longmapsto \phi'_a = \phi_a + X_a$ where $X_a = \delta\phi_a$

In order to preserve path: $X_a(\vec{x}_A, t_A) = X_a(\vec{x}_B, t_B) = 0$

The effect and the Action S and Lagrangian Density $\mathcal{L}$ are:

$S \longmapsto S'$ and $\mathcal{L} \longmapsto \mathcal{L}'$

For this transformation to be a symmetry of the action: $\delta S = \delta(S - S') = 0$

By looking at δS we see: $\delta S = \int d^4x\, \delta\mathcal{L}$

That is, $\delta S = 0$ if

- $\delta\mathcal{L} = 0$ (i.e. $\mathcal{L}$ is invariant)
- $\delta\mathcal{L} = \partial_\mu F^\mu$ (i.e $\mathcal{L}$ changes by a total derivative) as long as F^μ vanishes at endpoints of path

We saw when deriving the Euler-Lagrange equations that:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) = \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) = \partial_\mu F^\mu$$

where a implies the a^{th} field ϕ_a

If Euler-Lagrange equations are satisfied:

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a\right) = \partial_\mu F^\mu \implies \partial_\mu j^\mu = 0 \quad \text{if } j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}X_a(\phi) - F^\mu(\phi)$$

Sum over all fields due to repeated a index ?

Example: Transformation and Energy-Momentum Tensor

Consider infinitesimal translations such as the following: $x^\mu \longrightarrow (x')^\mu = x^\mu - \epsilon^\mu$ where $\epsilon^\mu = $ const i.e. Spatial and time translation

As x^μ is a contravariant quantity and as a result it transforms opposite to basis vectors

We can write $\delta x^\mu = \left(\frac{\partial x'^\mu}{\partial x^\mu}\right) = -\epsilon^\mu$ and $x^\mu = (x')^\mu + \epsilon^\mu$

Plus sign because new dependence of field is on $x^\mu + \epsilon^\mu$ i.e. $\delta x^\mu = \epsilon^\mu$ i.e. Active Transform

The field transforms as follows:

$$\phi_a(x^\mu) \longrightarrow \phi'_a(x^\mu) = \underbrace{\phi_a(x^\mu + \epsilon^\mu)}_{\text{old field in equivalent position to } x^\mu \text{ in new frame } x'} = \phi(x^\mu) + \frac{\partial\phi(x^\mu)}{\partial x^\mu}\delta x^\mu = \phi_a(x^\mu) + \epsilon^\mu\partial_\mu\phi_a(x^\mu)$$

i.e. $\phi_a(x^\mu) \longrightarrow \phi_a(x^\mu) + X_a(\phi)$ where $X_a(\phi) = \epsilon^\mu\partial_\mu\phi_a(x^\mu)$

How does the Lagrangian change?

$$\begin{aligned}
\mathcal{L}(\phi) \longrightarrow \mathcal{L}(\phi') &= \mathcal{L}(\phi) + (\mathcal{L}(\phi') - \mathcal{L}(\phi))(\Delta\phi/\Delta\phi) = \\
&= \mathcal{L}(\phi) + (\partial\mathcal{L}/\partial\phi)\delta\phi = \\
&= \mathcal{L}(\phi) + (\partial\mathcal{L}/\partial x^\mu)(\partial x^\mu/\partial\phi)\delta\phi = \\
&= \mathcal{L}(\phi) + (\partial_\mu\mathcal{L})(\partial_\mu\phi_a)^{-1}(\epsilon^\mu\partial_\mu\phi_a) = \\
&= \mathcal{L}(\phi) + \epsilon^\mu\partial_\mu\mathcal{L}
\end{aligned}$$

i.e. $\delta\mathcal{L} = \partial_\mu(\epsilon^\mu\mathcal{L}) = \partial_\mu F^\mu$ where $F^\mu = \epsilon^\mu\mathcal{L}$

Action is symmetric wrt traslations in space (Mom. is conserved) and in Time (energy is conserved)

Noether's Current:

Four conserved currents i.e. one for each component of ϵ^μ

$$(j^\nu)_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}X_a - F^\nu = \epsilon^\mu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\partial_\mu\phi_a - \epsilon^\nu\mathcal{L}$$

$$= \epsilon^\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_a)}\partial_\mu\phi_a - \delta^\nu_\mu\mathcal{L}\right] = \epsilon^\mu T^\nu_\mu \quad \text{Energy-Momentum Tensor}$$

As ϵ^μ is a const. T^ν_μ is also a conserved current i.e. $\partial_\nu T^\nu_\mu = 0$

The four conserved quantities are:

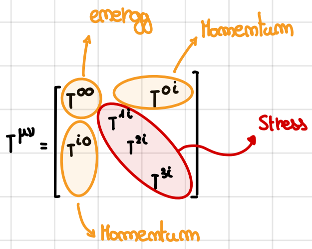
$$E = \int d^3x\, T^{00} \quad \text{and} \quad P^i = \int d^3x\, T^{0i}$$

An example of the Energy-Momentum Tensor

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \Longrightarrow T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

Using eq. of motion one can prove $\partial_\mu T^{\mu\nu} = 0$

E.O.M: $\Box\phi + m^2\phi = \ddot{\phi} - \nabla^2\phi + m^2\phi = 0$ i.e. $\Box\phi = -m^2\phi$



$$\begin{aligned} T^{\mu\nu} &= \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{1}{2}\eta^{\mu\nu}m^2\phi^2 = \\ &= \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\left[\partial^\beta\phi\partial_\beta\phi - m^2\phi^2\right] \\ &= \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}\eta^{\mu\nu}\left[\frac{1}{2}\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right] \end{aligned}$$

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= (\partial^2\phi)\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi) + \eta^{\mu\nu}(m^2\phi)\partial_\mu\phi - \frac{1}{2}\eta^{\mu\nu}\left[(\partial_\mu\partial^\beta\phi)\partial_\beta\phi + \partial_\beta\phi(\partial_\mu\partial_\beta\phi)\right] = \\ &= (\partial^2\phi)\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi) + m^2\phi\partial^\nu\phi - \partial_\beta\phi(\partial^\nu\partial_\beta\phi) = \\ &= \underbrace{\left[(\partial^2\phi) - \Box\phi\right]}_{0}\partial^\nu\phi + \partial^\mu\phi(\partial_\mu\partial^\nu\phi) - \partial_\beta\phi(\partial_\beta\partial^\nu\phi) = 0 \end{aligned}$$

Conserved Quantities:

Energy: $E = \int d^3x\, T^{00} = \frac{1}{2}\int d^3x\left[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right]$ ⟵ Time translation Symmetry

Momentum: $P^i = \int d^3x\, T^{0i} = \int d^3x\,\dot{\phi}\,\partial^i\phi$ ⟵ Spatial Translations Symmetry

} Space-time translation

N.B. In this example $T^{\mu\nu}$ is symmetric (i.e $T^{\mu\nu} = -T^{\mu\nu}$). However in some cases it isn't

Nevertheless we can add a new Tensor $\Gamma^{\rho\mu\nu}$ that is anti-symmetric w.r.t. exchange of the first two indices i.e. $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu}$

As a result $\partial_\mu\partial_\rho\Gamma^{\rho\mu\nu} = 0$ and $\partial_\mu\Theta^{\mu\nu} = 0$, where $\Theta^{\mu\nu}$ is the new E-M tensor $\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho\Gamma^{\rho\mu\nu}$

$$\partial_\mu\partial_\rho\Gamma^{\rho\mu\nu} = -\partial_\mu\partial_\rho\Gamma^{\mu\rho\nu} = -\partial_\rho\partial_\mu\Gamma^{\mu\rho\nu} = -\left[\partial_\mu\partial_\rho\Gamma^{\rho\mu\nu}\right] \Longrightarrow \partial_\mu\partial_\rho\Gamma^{\rho\mu\nu} = 0$$

e.g. General relativity in Flat Spacetime $\quad \Theta^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}\Big|_{g_{\mu\nu}=\eta_{\mu\nu}}$

Example: Lorentz Transform and Angular Momentum

What conserved quantity do Lorentz Transf. correspond to?

What is the equivalent of rotational symmetry?

Kronecker-Delta?

Infinitesimal form of Lorentz Transf.: $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$ where ω^μ_ν is infinitesimal

Condition for Lorentz Transf: $\Lambda^\mu_\alpha\eta^{\alpha\beta}\Lambda^\nu_\beta = \eta^{\mu\nu} \Longrightarrow (\delta^\mu_\sigma + \omega^\mu_\sigma)\eta^{\sigma\tau}(\delta^\nu_\tau + \omega^\nu_\tau) =$

$$\begin{aligned} &= \delta^\mu_\sigma\eta^{\sigma\tau}\delta^\nu_\tau + \delta^\mu_\sigma\eta^{\sigma\tau}\omega^\nu_\tau + \omega^\mu_\sigma\eta^{\sigma\tau}\delta^\nu_\tau + \omega^\mu_\sigma\eta^{\sigma\tau}\omega^\nu_\tau = \\ &= \eta^{\mu\nu} + \delta^\mu_\sigma\omega^{\nu\sigma} + \omega^\mu_\sigma\delta^{\nu\sigma} + \omega^\mu_\sigma\omega^{\nu\sigma} = \\ &= \eta^{\mu\nu} + \omega^{\nu\mu} + \omega^{\mu\nu} + \omega^\rho_\sigma\omega^{\nu\sigma} \end{aligned}$$

As ω is infinitesimal we have: $\omega^\mu_\sigma\omega^{\nu\sigma} \approx 0$

Then for $(\delta^\mu_\sigma + \omega^\mu_\sigma)\eta^{\sigma\tau}(\delta^\nu_\tau + \omega^\nu_\tau) = \eta^{\mu\nu}$ we need $\omega^{\nu\mu} + \omega^{\mu\nu} = 0$

$\omega^{\mu\nu}$ is anti-symmetric

There 6 anti-sym 4x4 matrices which is equal to the number of Lorentz Transf (3 boosts + 3 rotations)

As seen earlier, $\phi(x) \longmapsto \phi'(x) = \phi(\Lambda^{-1}x)$

As $\omega^{\mu\nu} = -\omega^{\nu\mu}$ we have $(\Lambda^{-1})^\mu_\nu = \delta^\mu_\nu - \omega^\mu_\nu$ and so $x^\nu \longmapsto x^\mu - \omega^\mu_\nu x^\nu$

↳ infinitesimal change

$\delta x = (x^\mu - \omega^\mu_\nu x^\nu) - x^\nu = -\omega^\mu_\nu x^\nu$

The change in the field is given by:

$$\phi' = \phi + (\phi' - \phi)(\Delta x/\Delta x) = \phi + \frac{\partial\phi}{\partial x^\mu}\delta x = \phi - \omega^\mu_\nu x^\nu\partial_\mu\phi \quad \text{or} \quad \delta\phi = -\omega^\mu_\nu x^\nu\partial_\mu\phi$$

The change in the Lagrangian is given by:

$$\mathcal{L} \longmapsto \mathcal{L}' \text{ and } \delta\mathcal{L} = \mathcal{L}' - \mathcal{L}$$

$$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L} = \mathcal{L} + (\mathcal{L}' - \mathcal{L})(\Delta x / \Delta x) =$$

$$= \mathcal{L} + \frac{\partial \mathcal{L}}{\partial x}\delta x = \mathcal{L} - \omega^{\mu}_{\ \nu} x^{\nu} (\partial_\mu \mathcal{L})$$

$$\delta\mathcal{L} = -\omega^{\mu}_{\ \nu} x^{\nu}(\partial_\mu \mathcal{L}) = \partial_\mu \left(-\omega^{\mu}_{\ \nu} x^{\nu} \mathcal{L}\right) \quad \text{as } \omega^{\mu}_{\ \mu} = 0 \text{ due to its anti-symmetry}$$

Noether's Current

$$\delta\phi = -\omega^{\mu}_{\ \nu} x^{\nu} \partial_\mu \phi$$

$$F^{\mu} = -\omega^{\mu}_{\ \nu} x^{\nu} \mathcal{L}$$

$$j^{\rho} = \omega^{\rho}_{\ \nu} x^{\nu} \mathcal{L} - \omega^{\mu}_{\ \nu} x^{\nu} \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} \partial_\mu \phi = -\omega^{\mu}_{\ \nu} T^{\rho}_{\ \mu} x^{\nu} \quad \text{where } T^{\rho}_{\ \mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi)} \partial_\mu \phi - \delta^{\rho}_{\mu} \mathcal{L}$$

While this a single current, we are more interested in the constituting currents $(j^{\rho})^{\mu\nu}$, one for each $\omega^{\mu}_{\ \nu}$

As $\omega^{\mu\nu} = -\omega^{\mu\nu}$ we have only 6 unique entries and thus currents

As $\partial_\rho (j^\rho)^{\mu\nu} = 0$ we can then strip away the $\omega^{\mu\nu}$ common factor

$(j^\rho)^{\mu\nu} = x^{\nu} T^{\rho}_{\ \mu}$ or $(\mathcal{J}^\rho)^{\mu\nu} = x^{\nu} T^{\rho}_{\ \mu} - x^{\mu} T^{\rho}_{\ \nu}$

The conserved current for each $(\mathcal{J}^\rho)^{\mu\nu}$ is given by $(\mathcal{J}^0)^{\mu\nu}$

As a result the conserved quantity is given by:

$$Q^{\mu\nu} = \int d^3x \left(x^{\nu} T^{0}_{\ \mu} - x^{\mu} T^{0}_{\ \nu}\right)$$

For $\mu, \nu = 1,2,3$ the Lorentz transformations are rotations $\Longrightarrow Q^{\mu\nu} = Q^{ij} \equiv$ Ang. Mom.

For $\mu = 0$ or $\nu = 0$ the Lorentz transf. are boosts $\Longrightarrow Q^{\mu\nu} = Q^{0i} = Q^{i0}$

What is Q^{0i}?

$$Q^{0i} = \int d^3x \left(x^0 T^{0i} - x^i T^{00}\right) \Longrightarrow \frac{dQ^{0i}}{dt} = 0 = \int d^3x\, T^{0i} + t\int d^3x \frac{\partial T^{0i}}{\partial t} - \frac{d}{dt}\int d^3x\, x^i T^{00} =$$

$$= P_i + t\frac{dP^i}{dt} - \frac{d}{dt}\int d^3x\, x^i T^{00}$$

As P_i is const. we have $\frac{d}{dt}\int d^3x\, x^i T^{00} = \text{const} \Longrightarrow$ Center of energy of field travels at constant velocity

Internal Symmetries

So far we have looked at transformations of spacetime and fields at the same time e.g. Lorentz Transform

However, there exists also "Internal Symmetries": Transformation of fields (and not of spacetime) which acts the same at every point in spacetime

Example: Field Rotation

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(|\varphi|^2)$$

$$\varphi \longrightarrow e^{i\alpha}\varphi \Longrightarrow \delta\varphi = i\alpha\varphi \quad \text{if } \alpha \ll 1$$

$$\varphi^* \longrightarrow e^{-i\alpha}\varphi^* \Longrightarrow \delta\varphi^* = -i\alpha\varphi \quad \text{if } \alpha \ll 1$$

$$\mathcal{L} \longrightarrow \mathcal{L}' = \overbrace{(e^{i\alpha})(e^{-i\alpha})}^{=1} \partial_\mu \varphi^* \partial^\mu \varphi - V(|\varphi|^2) = \mathcal{L} \qquad \delta\mathcal{L} = 0$$

Lagrangian is invariant under this transformation

The conserved current is then: $j^\mu = i(\partial^\mu \varphi^*)\varphi - i\varphi^*(\partial^\mu \varphi)$

Consider n Scalar Fields labeled by ϕ_a with the same mass.

The Lagrangian is then:

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{n} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}\sum_{a=1}^{n} m^2 \phi_a^2 - g\left(\sum_{a=1}^{n} \phi_a^2\right)^2$$

This Lagrangian is invariant under non-Abelian symmetry group $G = O(n)$ or $SO(n)$

For complex fields, we can construct Lagrangians that are invariant under $SU(n)$

Non-Abelian symmetries of this type are known as global symmetries

A cute trick

Consider an internal symmetry transformation of the kind $\delta\varphi = \alpha\varphi$ where $\alpha = \text{const}$.

We saw earlier that these transformations have $\delta\mathcal{L} = 0$

Now, $\alpha \longrightarrow \alpha(x)$ and $\delta\mathcal{L} = (\partial_\mu \alpha) h^\mu(\phi) = \partial_\mu(\alpha h^\mu) - \alpha\,\partial_\mu h^\mu$ such that $\delta\mathcal{L} = 0$ when $\alpha(x) = \text{const}$

Then:

$$\delta S = \int d^4x\, \delta\mathcal{L} = \int d^4x\, \partial_\mu(\alpha h^\mu) - \int d^4x\, \alpha\, \partial_\mu h^\mu = -\int d^4x\, \alpha(x)\, \partial_\mu h^\mu$$

As $\delta S = 0$, $\partial_\mu h^\mu = 0$ i.e. $h^\mu = j^\mu$

N.B. If you workout an example with $\alpha(x)$ you will see that only derivative terms contribute to h^μ

N.B. This works also for non-Abelian symmetries but $\alpha(x)$ is not a function but a matrix

Hamiltonian Formalism

Consider the scalar field(s) $\phi_a(x)$ with Lagrangian Density $\mathcal{L}(\phi_a, \dot{\phi}_a, \vec{\nabla}\phi) = \mathcal{L}(x)$ (as ϕ depends on x)

We define the Generalised Momentum conjugate to ϕ_a as $\pi^a(x) = (\partial\mathcal{L}/\partial\dot{\phi}_a)$

The Hamiltonian Density $\mathcal{H}$ is defined as follows: $\mathcal{H} = \pi^a(x)\dot{\phi}_a(x) - \mathcal{L}(x)$ such that $H = \int d^3x\, \mathcal{H}$

The equations of motion are given by:

$$\dot{\phi}(\vec{x},t) = \frac{\partial H}{\partial \pi(\vec{x},t)} \quad \text{and} \quad \dot{\pi}(\vec{x},t) = -\frac{\partial H}{\partial \phi(\vec{x},t)} \implies \text{Do not look Lorentz Invariant!}$$

N.B. While Lagrangian formalism is manifestly Lorentz invariant as Action is Lorentz Invariant, the Hamiltonian formalism is not as we have picked a preferred time. Nonetheless, all final answers must be Lorentz Invariant for a relativistic theory: we always have to check!

Example: A real Scalar Field

Consider $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi)$ for a real scalar field ϕ

The generalised momentum is $\pi(x) = \dot{\phi}$

Hamiltonian Density $\mathcal{H} = \pi(x)\dot{\phi} - \mathcal{L} = \pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)$

Hamiltonian: $H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi)\right]$

↳ As we saw earlier Energy is conserved for this system and now we can see that the Hamiltonian is equal to the total energy

Lecture 15.02.2024

Canonical Quantization

Canonical quantization: Process to go from generalised coordinates and momenta (i.e. Hamiltonian formalism) to quantum theory by promoting them to operators

e.g. Classical Dynamics: $q_a \longrightarrow \hat{q}_a$ and $p^a \longrightarrow \hat{p}^a$

Must satisfy: $[q_a, q_b] = [p^a, p^b] = 0$ and $\boxed{[q_a, p^b] = i\delta_a^b}$

e.g. Fields: $\phi(\vec{x}) \longrightarrow \hat{\phi}(\vec{x})$, $\pi(\vec{x}) \longrightarrow \hat{\pi}(\vec{x})$

Must satisfy: $[\phi_a, \phi_b] = [\pi^a, \pi^b] = 0$

$\boxed{[\phi_a(\vec{x}), \pi^b(\vec{y})] = i\underbrace{\delta^3(\vec{x}-\vec{y})}_{}\delta_a^b}$

↳ field at two diff. points

→ These two determine the quantization

N.B. By allowing ϕ and π to become operator we have separated $\vec{x}$ and t and thus lost track of Lorentz Invariance. All time dependence sits in the states $|\psi\rangle$ whic evolve according to $i\frac{d|\psi\rangle}{dt} = H|\psi\rangle$

N.B. $|\psi\rangle$ is a functional i.e. a function containing all possible field configurations

These configurations are many as the fields have infinite degrees of freedom

$|\psi\rangle$ is acted upon by ϕ, π

Why are fields important? They provide the correlation between two different spacetime points from which everything can be derived

Free Theories

Determining spectrum of H is typically very hard as there are infinite degrees of freedom

However, in free theories one can write the dynamics of a system such that all d.o.f. evolve independently

↳ Free theories have generally Quadratic Lagrangians and linear e.o.m

Example: Classical KG equation

Classical Klein-Gordon e.o.m for scalar field $\phi(\vec{x},t)$: $\partial_\mu\partial^\mu\phi + m^2\phi = 0$

We can decouple degrees of freedom through Fourier transf: $\phi(\vec{x},t) = \frac{1}{(2\pi)^3}\int d^3p\, e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p},t)$

The equation becomes: $\left(\frac{\partial^2}{\partial t^2} + (\vec{p}^2 + m^2)\right)\phi(\vec{p},t) = \ddot{\phi}(\vec{p},t) + \omega_p^2\,\phi(\vec{p},t) = 0$

This equation is equivalent to a harmonic oscillator with ang. freq. $\omega_p = \sqrt{\vec{p}^2+m^2}$ i.e. dispersion relationship

The most general solution to KG equation is an infinite superposition of harmonic oscillators with different momentum

As $\phi(\vec{p},t)$ is a harmonic oscillator $\forall\vec{p}$, to quantize $\phi(\vec{x},t)$ we must quantize the infinite number of harmonic oscillators

The Simple Harmonic Oscillator

Potential energy: $U(q) = \frac{1}{2}Kq^2 = \frac{1}{2}m\,\omega^2 x^2$

Kinetic energy: $K(p) = \frac{1}{2}m\,\dot{x}^2$

Hamiltonian: $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$ where $q = mx^2$ and $p = \dot{q}$ with $[q,p] = i$

We now define the raising and lowering operators or "creation" and "annihilation" operators: $a = (\sqrt{\omega/2})q + i(\sqrt{2\omega})^{-1}p$, $a^\dagger = (\sqrt{\omega/2})q - i(\sqrt{2\omega})^{-1}p$

We can write: $q = (\sqrt{2\omega})^{-1}(a + a^\dagger)$ and $q = -i(\sqrt{\omega/2})(a - a^\dagger)$

Exploiting these relations we get: $[a, a^\dagger] = 1$ as $[q,p] = qp - pq = -i(1/2)\left[a^2 - (a^\dagger)^2 - aa^\dagger + a^\dagger a - a^2 + (a^\dagger)^2 - aa^\dagger + a^\dagger a\right] = i[aa^\dagger - a^\dagger a] = i[a, a^\dagger] = i$

Then: $q^2 = (2\omega)^{-1}\left[a^2 + (a^\dagger)^2 + aa^\dagger + a^\dagger a\right]$ $\quad p^2 = -(\omega/2)\left[a^2 + (a^\dagger)^2 - aa^\dagger - a^\dagger a\right]$

$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \frac{1}{2}\omega\left([a,a^\dagger] + 2a^\dagger a\right) = \omega\left(a^\dagger a + \frac{1}{2}\right)$

$[H, a^\dagger] = Ha^\dagger - a^\dagger H = \omega\left[a^\dagger(aa^\dagger) + \frac{1}{2}a^\dagger - (a^\dagger)^2 a - \frac{1}{2}a^\dagger\right] = \omega a^\dagger\left[aa^\dagger - a^\dagger a\right] = \omega a^\dagger$

$[H, a] = Ha - aH = \omega\left[a^\dagger(a)^2 + \frac{1}{2}a - aa^\dagger a - \frac{1}{2}a\right] = \omega[a^\dagger a - aa^\dagger]a = -\omega a$

Look at inverted harmonic oscillator

Let $|E\rangle$ be an eigenstate of H such that $H|E\rangle = E|E\rangle$.

Then: $Ha|E\rangle = ([H,a] + aH)|E\rangle = (E-\omega)a|E\rangle = H|E-\omega\rangle$ where $|E-\omega\rangle = a|E\rangle$ i.e. annihilation op

$Ha^\dagger|E\rangle = ([H,a^\dagger] + a^\dagger H)|E\rangle = (E+\omega)a^\dagger|E\rangle = H|E+\omega\rangle$ where $|E+\omega\rangle = a^\dagger|E\rangle$ i.e. creation op.

If energy is bound from below we have a ground state $|E_0\rangle$ such that $|E_n\rangle = (a^\dagger)^n|E_0\rangle$ and $H|E_0\rangle = \frac{1}{2}\omega|E_0\rangle$ and $H|E_n\rangle = (n + 1/2)|E_n\rangle$

Be aware that $|E_n\rangle$ states are not yet normalized i.e. $\langle E_n|E_n\rangle \neq 1$

The Free Scalar Field

We want to apply these concepts to $\phi(\vec{x})$ and $\Pi(\vec{x})$

The solutions to the classical KG equation are two plane wave solutions: $\phi(\vec{p},t) = Ae^{i\omega_p t} + Be^{-i\omega_p t}$ (Where did the mass go?)

We can thus write $\phi(\vec{x}), \Pi(\vec{x})$ as an infinite sum of harmonic oscillator states as follows:

$$\phi(\vec{x}) = \int d^3p\,(2\pi)^{-3}\, q\, e^{i\vec{p}\cdot\vec{x}} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$\Pi(\vec{x}) = \int d^3p\,(2\pi)^{-3}\, p\, e^{i\vec{p}\cdot\vec{x}} = \int \frac{d^3p}{(2\pi)^3}(-i)\left(\sqrt{\omega_{\vec{p}}/2}\right)\left[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]$$

Note: $\delta^3(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$

Equivalence of commutators

Claim

$$\begin{aligned}[\phi_a(\vec{x}), \phi_b(\vec{y})] &= [\Pi^a(\vec{x}), \Pi^b(\vec{y})] = 0 \\ [\phi_a(\vec{x}), \Pi^b(\vec{y})] &= i\,\delta(\vec{x}-\vec{y})\,\delta^a_b\end{aligned} \iff \begin{aligned}[{}^a a_{\vec{p}}, {}^b a_{\vec{q}}] &= [{}^a a^\dagger_{\vec{p}}, {}^b a^\dagger_{\vec{q}}] = 0 \\ [{}^a a_{\vec{p}}, {}^b a^\dagger_{\vec{q}}] &= (2\pi)^3\,\delta(\vec{p}-\vec{q})\,\delta^a_b\end{aligned}$$

We have to show this claim holds from left to right and from right to left

Right $\longrightarrow$ Left

We can derive some commutator relationships from their definition

$[{}^a a_{\vec{p}}, {}^b a_{\vec{q}}] = -[{}^b a_{\vec{q}}, {}^a a_{\vec{p}}] = a_{\vec{p}}a_{\vec{q}} - a_{\vec{q}}a_{\vec{p}}$

$[{}^a a^\dagger_{\vec{p}}, {}^b a^\dagger_{\vec{q}}] = -[{}^b a^\dagger_{\vec{q}}, {}^a a^\dagger_{\vec{p}}] = a^\dagger_{\vec{p}}a^\dagger_{\vec{q}} - a^\dagger_{\vec{q}}a^\dagger_{\vec{p}}$

As "a","b" and "$\vec{p}$","$\vec{q}$" are just labels and the involved operators are the same, switching them around does not change commutato. Then: $[{}^a a_{\vec{p}}, {}^b a_{\vec{q}}] = -[{}^a a_{\vec{p}}, {}^b a_{\vec{q}}]$ i.e. $[{}^a a_{\vec{p}}, {}^b a_{\vec{q}}] = [{}^a a^\dagger_{\vec{p}}, {}^b a^\dagger_{\vec{q}}] = 0$

$[{}^a a_{\vec{p}}, {}^b a^\dagger_{\vec{q}}] = -[{}^b a^\dagger_{\vec{q}}, {}^a a_{\vec{p}}] = a_{\vec{p}}a^\dagger_{\vec{q}} - a^\dagger_{\vec{q}}a_{\vec{p}} \implies$ As operators are different, switching labels might affect commutator

Then, $[{}^a a_{\vec{p}}, {}^b a_{\vec{q}}] = [{}^a a^\dagger_{\vec{p}}, {}^b a^\dagger_{\vec{q}}] = 0$ which holds with our claim

Now let's check whether $[{}^a a_{\vec{p}}, {}^b a^\dagger_{\vec{q}}] = (2\pi)^3\,\delta(\vec{p}-\vec{q})\,\delta^a_b$ holds by explicitely computing $[\phi_a(\vec{x}), \Pi_b(\vec{y})]$

$$[\phi_a(\vec{x}), \Pi^b(\vec{y})] = \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2i}\sqrt{\frac{\omega_q}{\omega_p}}\left\{\left[{}^a a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + {}^a a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]\left[{}^b a_{\vec{q}}e^{i\vec{q}\cdot\vec{y}} - {}^b a^\dagger_q e^{-i\vec{q}\cdot\vec{y}}\right] - \left[{}^b a_{\vec{q}}e^{i\vec{q}\cdot\vec{y}} - {}^b a^\dagger_q e^{-i\vec{q}\cdot\vec{y}}\right]\left[{}^a a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + {}^a a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]\right\} =$$

$$= \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2i}\sqrt{\frac{\omega_q}{\omega_p}}\left\{[{}^a a_{\vec{p}}, {}^b a_{\vec{q}}]\exp[i(\vec{p}\cdot\vec{x}) + i(\vec{q}\cdot\vec{y})] - [{}^a a_{\vec{p}}, {}^b a^\dagger_{\vec{q}}]\exp[i(\vec{p}\cdot\vec{x}) - i(\vec{q}\cdot\vec{y})] + \right.$$

$$\left. + [{}^a a^\dagger_{\vec{p}}, {}^b a_{\vec{q}}]\exp[-i(\vec{p}\cdot\vec{x}) + i(\vec{q}\cdot\vec{y})] - [{}^a a^\dagger_{\vec{p}}, {}^b a^\dagger_q]\exp[-i(\vec{p}\cdot\vec{x}) - i(\vec{q}\cdot\vec{y})]\right\} =$$

$$= \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2i}\sqrt{\frac{\omega_q}{\omega_p}}\left\{[{}^a a^\dagger_{\vec{p}}, {}^b a_{\vec{q}}]\,e^{i(\vec{q}\cdot\vec{y} - \vec{p}\cdot\vec{x})} - [{}^a a_{\vec{p}}, {}^b a^\dagger_{\vec{q}}]\,e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})}\right\} =$$

$$= \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{i(2\pi)^3}{2}\sqrt{\frac{\omega_q}{\omega_p}}\left\{\delta(\vec{p}-\vec{q})\delta^a_b e^{i(\vec{q}\cdot\vec{y} - \vec{p}\cdot\vec{x})} + \delta(\vec{p}-\vec{q})\delta^a_b e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})}\right\} =$$

$$= \delta^a_b\int\frac{d^3p}{(2\pi)^3}\frac{i}{2}\left[e^{i\vec{p}(\vec{x}-\vec{y})} + e^{i\vec{p}\cdot(\vec{y}-\vec{x})}\right] = \frac{i}{2}\delta^a_b\left[\delta(\vec{x}-\vec{y}) + \delta(\vec{y}-\vec{x})\right] = i\,\delta^a_b\,\delta(\vec{x}-\vec{y})$$

We have proven that the above claim hold from right to left.

Left $\longrightarrow$ Right

With labels argument we can easily show: $[\phi_a(\vec{x}), \phi_b(\vec{y})] = [\Pi^a(\vec{x}), \Pi^b(\vec{y})]$

We now have to prove that $[{}^a a_{\vec{p}}, {}^b a^\dagger_{\vec{q}}] = (2\pi)^3\,\delta(\vec{p}-\vec{q})\,\delta^a_b$ using $[\phi_a(\vec{x}), \Pi^b(\vec{y})] = i\,\delta(\vec{x}-\vec{y})\,\delta^a_b$

For this we need an expression of $a_{\vec{p}}, a^\dagger_{\vec{p}}$ in terms of $\phi(\vec{x}), \Pi(\vec{x})$

We have that:

renaming to make it shorter

$\phi_p(\vec{x}) = (d/d\vec{p})\,\phi(\vec{x}) = (2\pi)^{-3}\left(\sqrt{2\omega_{\vec{p}}}\right)^{-1}\left[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]$ and $\Pi_p(\vec{x}) = (d/d\vec{p})\,\Pi(x) = (2\pi)^{-3}(-i)\left(\sqrt{\omega_{\vec{p}}/2}\right)\left[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}'} - a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]$

$a_{\vec{p}} = (1/2)(2\pi)^3\left(\sqrt{2\omega_p}\right)\left[\phi_p(\vec{x}) + i\,\omega_{\vec{p}}^{-1}\,\Pi_p(\vec{x})\right]e^{-i\vec{p}\cdot\vec{x}}$

$a^\dagger_{\vec{p}} = (1/2)(2\pi)^3\left(\sqrt{2\omega_p}\right)\left[\phi_p(\vec{x}) - i\,\omega_{\vec{p}}^{-1}\,\Pi_p(\vec{x})\right]e^{+i\vec{p}\cdot\vec{x}}$

Left $\longrightarrow$ Right

Let's look at the structure of $\phi_{\vec p}$ and Π_p

$$\phi_p = (2\pi)^{-3}\phi(\vec p)\, e^{i\vec p\cdot\vec x} \quad \text{where} \quad \phi(\vec p) = \int d^3x\, \phi(\vec x) e^{-i\vec p\cdot\vec x} \Longrightarrow \phi_p = e^{i\vec p\cdot\vec x}\int \frac{d^3x}{(2\pi)^3}\phi(\vec x)\, e^{-i\vec p\cdot\vec x}$$

$$\Pi_p = (2\pi)^{-3}\Pi(\vec p)\, e^{i\vec p\cdot\vec x} \quad \text{where} \quad \Pi(\vec p) = \int d^3x\, \Pi(\vec x) e^{-i\vec p\cdot\vec x} \Longrightarrow \Pi_p = e^{i\vec p\cdot\vec x}\int \frac{d^3x}{(2\pi)^3}\Pi(\vec x)\, e^{-i\vec p\cdot\vec x}$$

Then we can write them down as:

$$a_{\vec p} = (1/2)(2\pi)^3(\sqrt{2\omega_p})\left[\phi_p(\vec x) + i\,\omega_p^{-1}\Pi_p(\vec x)\right]e^{-i\vec p\cdot\vec x} \Longrightarrow a_{\vec p} = \int d^3x\left[\sqrt{\frac{\omega_p}{2}}\phi(\vec x) + \frac{i}{\sqrt{2\omega_p}}\Pi(\vec x)\right]e^{-i\vec p\cdot\vec x}$$

$$a^\dagger_{\vec p} = (1/2)(2\pi)^3(\sqrt{2\omega_p})\left[\phi_p(\vec x) - i\,\omega_p^{-1}\Pi_p(\vec x)\right]e^{+i\vec p\cdot\vec x} \Longrightarrow a^\dagger_{\vec p} = e^{i2\vec p\cdot\vec x}\int d^3x\left[\sqrt{\frac{\omega_p}{2}}\phi(\vec x) - \frac{i}{\sqrt{2\omega_p}}\Pi(\vec x)\right]e^{-i\vec p\cdot\vec x}$$

As $e^{i2\vec p\cdot\vec x} = \int d^3y\, \delta(\vec x-\vec y)\, e^{i2\vec p\cdot\vec y}$ we have:

$$a_{\vec p} = \int d^3x\left[\sqrt{\frac{\omega_p}{2}}\phi(\vec x) + \frac{i}{\sqrt{2\omega_p}}\Pi(\vec x)\right]e^{-i\vec p\cdot\vec x}$$

$$a^\dagger_{\vec p} = \int d^3x\left[\sqrt{\frac{\omega_p}{2}}\phi(\vec x) - \frac{i}{\sqrt{2\omega_p}}\Pi(\vec x)\right]e^{-i\vec p\cdot\vec x}$$

$$[{}^a a_{\vec p}, {}^b a^\dagger_q] = \int d^3x\, d^3y\left\{\frac12\sqrt{\omega_p\omega_q}\,[\phi_a(\vec x),\phi_b(\vec y)]e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec y)} - \frac{i}{2}\sqrt{\frac{\omega_p}{\omega_q}}[\phi_a(\vec x),\Pi^b(\vec y)]e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec y)} + \frac{i}{2}\sqrt{\frac{\omega_q}{\omega_p}}[\Pi_a(\vec x),\phi_b(\vec y)]e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec y)} + \frac{1}{2\sqrt{\omega_p\omega_q}}[\Pi_a(\vec x),\Pi_b(\vec y)]\right\}$$

$$= \int d^3x\, d^3y\left(\frac{i}{2}\right)\left[-\sqrt{\frac{\omega_p}{\omega_q}}[\phi_a(\vec x),\Pi^b(\vec y)]e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec y)} + \sqrt{\frac{\omega_q}{\omega_p}}[\Pi_a(\vec x),\phi_b(\vec y)]e^{-(i\vec p\cdot\vec x-i\vec q\cdot\vec y)}\right\}$$

$$= \int d^3x\, d^3y\left(\frac{1}{2i}\right)\left[i\,\delta(\vec x-\vec y)\delta^a_b\, e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec y)} + i\,\delta(\vec y-\vec x)\delta^a_b\, e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec y)}\right\}$$

$$= \int d^3x\, e^{i(\vec p-\vec q)\cdot\vec x}\,\delta^a_b = (2\pi)^3\delta(\vec x-\vec y)\delta^a_b$$

Similarly, exploiting $[\phi_a(x),\phi_b(y)]=0$ we can prove $[a_{\vec p}, a_{\vec q}]=0$

The Hamiltonian

For the Lagrangian density $\mathcal{L} = \frac12\partial_\mu\phi\partial^\mu\phi - V(\phi)$ the Hamiltonian is given by $H = \int d^3x\left[\frac12\Pi^2 + \frac12(\vec\nabla\phi)^2 + V(\phi)\right]$

We also have that:

$$\phi(\vec x) = \int\frac{d^3p}{(2\pi)^3}(\sqrt{2\omega_p})^{-1}\left[a_{\vec p}e^{i\vec p\cdot\vec x} + a^\dagger_p e^{-i\vec p\cdot\vec x}\right]$$

$$\Pi(\vec x) = \int\frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left[a_{\vec p}e^{i\vec p\cdot\vec x} - a^\dagger_{\vec p}e^{-i\vec p\cdot\vec x}\right]$$

$$\vec\nabla\phi(\vec x) = \int\frac{d^3p}{(2\pi)^3}i(\sqrt{2\omega_p})^{-1}\left[a_{\vec p}\vec p\, e^{i\vec p\cdot\vec x} - a^\dagger_{\vec p}\vec p\, e^{-i\vec p\cdot\vec x}\right]$$

$$\Pi^2(\vec x) = \int\frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left[a_{\vec p}e^{i\vec p\cdot\vec x} - a^\dagger_{\vec p}e^{-i\vec p\cdot\vec x}\right]\int\frac{d^3q}{(2\pi)^3}(-i)\sqrt{\frac{\omega_q}{2}}\left[a_{\vec q}e^{i\vec q\cdot\vec x} - a^\dagger_{\vec q}e^{-i\vec q\cdot\vec x}\right]$$

$$= \int\frac{d^3p\, d^3q}{(2\pi)^6}\frac{(-1)}{2}\sqrt{\omega_p\omega_q}\left[a_p a_q\, e^{i(\vec p\cdot\vec x+\vec q\cdot\vec x)} - a_{\vec p}a^\dagger_{\vec q}\, e^{i\vec p\cdot\vec x-i\vec q\cdot\vec x} - a^\dagger_{\vec p}a_{\vec q}\, e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec x)} + a^\dagger_{\vec p}a^\dagger_{\vec q}\, e^{i(\vec p\cdot\vec x+\vec q\cdot\vec x)}\right]$$

$$(\vec\nabla\phi)^2 = \int\frac{d^3p}{(2\pi)^3}i(\sqrt{2\omega_p})^{-1}\left[a_{\vec p}\vec p\, e^{i\vec p\cdot\vec x} - a^\dagger_{\vec p}\vec p\, e^{-i\vec p\cdot\vec x}\right]\int\frac{d^3q}{(2\pi)^3}i(\sqrt{2\omega_p})^{-1}\left[a_{\vec q}\vec q\, e^{i\vec q\cdot\vec x} - a^\dagger_{\vec q}\vec q\, e^{-i\vec q\cdot\vec x}\right] =$$

$$= \int\frac{d^3p}{(2\pi)^6}(i)^2\frac{\vec p\cdot\vec q}{2\sqrt{\omega_p\omega_q}}\left[a_p a_q\, e^{i(\vec p\cdot\vec x+\vec q\cdot\vec x)} - a_{\vec p}a^\dagger_{\vec q}\, e^{i\vec p\cdot\vec x-i\vec q\cdot\vec x} - a^\dagger_{\vec p}a_{\vec q}\, e^{-i(\vec p\cdot\vec x-\vec q\cdot\vec x)} + a^\dagger_{\vec p}a^\dagger_{\vec q}\, e^{i(\vec p\cdot\vec x+\vec q\cdot\vec x)}\right]$$

If $V(\phi) = m^2\phi^2$ we have:

$$\phi^2 = \int\frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{2\sqrt{\omega_p\omega_q}}\left[a_{\vec p}a_{\vec q}\, e^{i(\vec p\cdot\vec x+\vec q\cdot\vec x)} + a_p a^\dagger_q\, e^{i(\vec p-\vec q)\cdot\vec x} + a^\dagger_{\vec p}a_{\vec q}\, e^{-i(\vec p-\vec q)\cdot\vec x} + a^\dagger_{\vec p}a^\dagger_{\vec q}\, e^{-i(\vec p+\vec q)\cdot\vec x}\right]$$

Then:

$$\Pi^2(\vec x) + (\vec\nabla\phi)^2 = -\int\frac{d^3p\, d^3q}{(2\pi)^6}\left[\sqrt{\omega_p\omega_q} + \frac{\vec p\cdot\vec q}{\sqrt{\omega_p\omega_q}}\right]\left[a_{\vec p}a_{\vec q}\, e^{i(\vec p+\vec q)\cdot\vec x} + a^\dagger_{\vec p}a^\dagger_{\vec q}\, e^{-i(\vec p+\vec q)\cdot\vec x} - a_{\vec p}a^\dagger_{\vec q}\, e^{i(\vec p-\vec q)\cdot\vec x} - a^\dagger_{\vec p}a_{\vec q}\, e^{-i(\vec p-\vec q)\cdot\vec x}\right] =$$

$$= -\int\frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{\omega_p\omega_q}}\left[\omega_p\omega_q + \vec p\cdot\vec q\right]\left\{\left[([a_{\vec p},a_{\vec q}] + a_{\vec q}a_{\vec p})e^{i(\vec p+\vec q)\cdot\vec x} + ([a^\dagger_{\vec p},a^\dagger_{\vec q}] + a^\dagger_{\vec q}a^\dagger_{\vec p})e^{-i(\vec p+\vec q)\cdot\vec x}\right] + \right.$$

$$\left. - \left[([a_{\vec p},a^\dagger_{\vec q}] + a^\dagger_{\vec q}a_{\vec p})e^{i(\vec p-\vec q)\cdot\vec x} + ([a^\dagger_{\vec p},a_{\vec q}] + a_{\vec q}a^\dagger_{\vec p})e^{-i(\vec p-\vec q)\cdot\vec x}\right]\right\}$$

As a result:

$$\frac{1}{2}\left[\pi^2(x)+(\vec{\nabla}\phi)^2\right]+V(\phi)=\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{\omega_p\omega_q}}\left[m^2+\omega_p\omega_q+\vec{p}\cdot\vec{q}\right]\Big\{\Big[\left(a_{\vec{p}}a_{\vec{q}}-a_{\vec{q}}a_{\vec{p}}-[a_{\vec{p}},a_{\vec{q}}]\right)e^{i(\vec{p}+\vec{q})\cdot\vec{x}}+\left(a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}-a^\dagger_{\vec{q}}a^\dagger_{\vec{p}}-[a^\dagger_{\vec{p}},a^\dagger_{\vec{q}}]\right)e^{-i(\vec{p}+\vec{q})\cdot\vec{x}}\Big]+$$

$$+\Big[\left(a^\dagger_{\vec{p}}a_{\vec{q}}+a_{\vec{q}}a^\dagger_{\vec{p}}+[a^\dagger_{\vec{p}},a_{\vec{q}}]\right)e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}+\left(a_{\vec{p}}a^\dagger_{\vec{q}}+a^\dagger_{\vec{q}}a_{\vec{p}}+[a_{\vec{p}},a^\dagger_{\vec{q}}]\right)e^{i(\vec{p}-\vec{q})\cdot\vec{x}}\Big]\Big\}=$$

$$=\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{\left[m^2+\omega_p\omega_q+\vec{p}\cdot\vec{q}\right]}{2\sqrt{\omega_p\omega_q}}\left\{2[a_{\vec{p}},a_{\vec{q}}]e^{i(\vec{p}+\vec{q})\cdot\vec{x}}+2[a^\dagger_{\vec{p}},a^\dagger_{\vec{q}}]e^{-i(\vec{p}+\vec{q})\cdot\vec{x}}+2a_pa^\dagger_{\vec{q}}e^{i(\vec{p}-\vec{q})\cdot\vec{x}}+2a^\dagger_{\vec{p}}a_{\vec{q}}e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}\right\}$$

$$=\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{\left[m^2+\omega_p\omega_q+\vec{p}\cdot\vec{q}\right]}{\sqrt{\omega_p\omega_q}}\left[a_{\vec{p}}a^\dagger_{\vec{q}}e^{i(\vec{p}-\vec{q})\cdot\vec{x}}+a^\dagger_{\vec{p}}a_{\vec{q}}e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}\right]$$

The Hamiltonian then becomes:

$$H=\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6}\frac{\left[m^2-\omega_p\omega_q-\vec{p}\cdot\vec{q}\right]}{\sqrt{\omega_p\omega_q}}\left[a_{\vec{p}}a^\dagger_{\vec{q}}e^{i(\vec{p}-\vec{q})\cdot\vec{x}}+a^\dagger_{\vec{p}}a_{\vec{q}}e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}\right]=$$

$$=\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{\left[m^2+\omega_p\omega_q+\vec{p}\cdot\vec{q}\right]}{\sqrt{\omega_p\omega_q}}\left[a_{\vec{p}}a^\dagger_{\vec{q}}(2\pi)^3\delta(\vec{p}-\vec{q})+a^\dagger_{\vec{p}}a_{\vec{q}}(2\pi)^3\delta(\vec{p}-\vec{q})\right]=$$

$$=\int\frac{d^3p}{(2\pi)^3}\frac{\left[m^2+\omega_p^2+p^2\right]}{\omega_p}\left[a_{\vec{p}}a^\dagger_{\vec{p}}+a^\dagger_{\vec{p}}a_{\vec{p}}\right]=$$

$$=\int\frac{d^3p}{(2\pi)^3}\frac{\left[m^2+\omega_p^2+p^2\right]}{\omega_p}\left[2a^\dagger_{\vec{p}}a_{\vec{p}}+[a_{\vec{p}},a^\dagger_p]\right]=$$

$$=\int\frac{d^3p}{(2\pi)^3}\frac{2(m^2+\omega_p^2+p^2)}{\omega_p}\left[a^\dagger_{\vec{p}}a_{\vec{p}}+(2\pi)^3\delta(0)\right]$$

$$=\int\frac{d^3p}{(2\pi)^3}(4\omega_p)\left[a^\dagger_{\vec{p}}a_{\vec{p}}+(2\pi)^3\delta(0)\right]$$ I forgot a factor of $1/4$?

Mistake

The Hamiltonian $H=\int\frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left[a^\dagger_{\vec{p}}a_{\vec{p}}+\frac{1}{2}(2\pi)^3\delta(0)\right]$ not only has a delta function but it also diverges as $\vec{p}\longrightarrow\infty$, what to do?

This is a delta function that evaluate at zero (where it is ∞) $\forall\,\vec{p}$

Vacuum

Assume that energy eigenstates are bounded from below by "vacuum" state $|0\rangle$ with energy eigenvalue E_0 such that $a_{\vec{p}}|0\rangle \ \forall \vec{p}$.
The energy can be computed b means of the Hamiltonian operator:

$$H|0\rangle = E_0|0\rangle = \left[\int \frac{d^3p}{(2\pi)^3}\left[a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}(2\pi)^3\delta^3(0)\right]\right]|0\rangle =$$

$$= \int \frac{d^3p}{(2\pi)^3}\left[a^\dagger_{\vec{p}} a_{\vec{p}}\right]|0\rangle + \frac{1}{2}\left[\int d^3p\, \omega_{\vec{p}}\, \delta^3(0)\right]|0\rangle =$$

$$= \frac{1}{2}\left[\int d^3p\, \omega_{\vec{p}}\, \delta^3(0)\right]|0\rangle = \infty|0\rangle \implies E_0 = \infty$$

In the above expression there actually two infinities present:

- Infra-red divergences arising due to the infinity of space (i.e Long wavelength divergence)
  - ↳ Consider the volume of a box of sides L: $V = \int_{-L/2}^{L/2} d^3x$ and when $L\to\infty$ $V = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x\, e^{i\vec{p}\cdot\vec{x}}\Big|_{\vec{p}=0} = (2\pi)^3\delta(0) = \infty$
  - ↳ We can adjust for this by computing energy density $\mathcal{E}_0 = E_0/V = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\omega_{\vec{p}}$
- Ultra-violet divergence arising due to the break-down of our theory at high $\vec{p}$ (i.e. Short distances, high frequencies)
  - ↳ Manifests as $\mathcal{E}_0 \to \infty$ as $|\vec{p}| \to \infty$

There's a way to deal with this infinities by considering that in physics we only measure energy differences (i.e. we do not measure E_0 directly)
We can thus remap the Hamiltonian $H|\psi\rangle \longrightarrow H|\psi\rangle - E_0$ i.e. by taking vacuum as reference
The Hamiltonian thus becomes: $H = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}}$

Normal Ordering $\longrightarrow$ Useful to extract finite part of infinities

The above Hamiltonian is merely the result of an ordering ambiguity that arises in the quantization of classical theories
e.g. $H = (1/2)(\omega q - ip)(\omega q + ip)$ upon quantization naturally gives $H = \omega a^\dagger a$

We define a string of operators $\phi_1(\vec{x}_1)\dots\phi_n(\vec{x}_n)$ to be normal ordered when all annihilation operators are to the right while all creation operators are to the left e.g. $H := \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}}$

Example: Cosmological Constant See Tong

Example: Casimir Effect

Using the normal ordering prescription E_0 can be set to $E_0 = 0$. However, in some situations we are interested in measuring differences in fluctuations of the vacuum energy. This is the case of the Casimir Effect

To consider this effect we can consider a massless scalar field $\phi(\vec{x})$ on which we impose the boundary conditions $\phi(\vec{x}) = \phi(\vec{x} + L\hat{x})$. This allows us to ignore the infrared divergence coming from the $\hat{x}$ direction as its size is restricted to L and thus momentum p_x is quantized. As y and z are unaffected, energies and other related quatities must be computed per unit area

We will now consider the situation in which two parallel planes separated by distance $d \ll L$ in $\hat{x}$ are embedded in the scalar field ϕ such that $\phi(x_1) = \phi(x_2) = 0$ where x_1 and x_2 are the locations along $\hat{x}$ of the two planes

Inside the planes:

The net effect on momentum is the following: $\vec{p} = (n\pi/d, p_y, p_z)$, $n \in \mathbb{Z}^+$
As we are dealing with a massless scalar field:

$$\omega_{\vec{p}} = |\vec{p}| = \sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2} \quad \text{and} \quad H = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left[a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}(2\pi)^3\delta^3(0)\right] \longmapsto H = \sum_{n=1}^{\infty}\int \frac{dp_y dp_z}{(2\pi)^2}\omega_{\vec{p}}\left[a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2}(2\pi)^2\delta^2(0)\right]$$

We are interested in the vacuum energy i.e. $E_0(d) = \sum_{n=1}^{\infty}\int \frac{dp_y dp_z}{(2\pi)^2}\frac{1}{2}\omega_{\vec{p}}(n)\left[(2\pi)^2\delta(0)\right]$
As $A = \lim_{\ell\to\infty}\int_{-\ell/2}^{\ell/2} dy\,dz = \lim_{\ell\to\infty}\int_{-\ell/2}^{\ell/2} e^{i\vec{p}\cdot\vec{x}}\Big|_{\vec{p}=0} dy\,dz = (2\pi)^2\delta(0)$ we have $\mathcal{E}_0(d) = E_0(d)/A = \sum_{n=1}^{\infty}\int \frac{dp_y dp_z}{(2\pi)^2}\frac{1}{2}\sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}$

Then:

- Energy inside the planes: $E(d)$
- Energy outside the planes: $E(L-d)$
- Total energy: $E = E(d) + E(L-d)$ $\Longrightarrow$ If E depends on d, vacuum energy has fluctuations and thus there is a force on the plates (Casimir Force)

The dependence of E on d is impossible to find as E is infinite due to the UV-divergence. However, one can realise that high momentum/frequency waves cannot be reflected by the planes as some parts of the wave would go through. We focus on completely reflected waves by introducing the UV cutoff wavelength a such that $a \ll d$. We artificially manipulate the integral as follows:

$$\varepsilon_0(d) = \sum_{n=1}^{\infty} \int \frac{dp_x dp_z}{(2\pi)^2} \frac{1}{2} \omega_{\vec{p}}(n)\, e^{-a\,\omega_{\vec{p}}(n)}$$ so that if $a \to 0$ we regain the original expression but if $a > 0$, the integral becomes finite by cutting $p \gg a^{-1}$

In order to have a meaningful result, a should not appear in the final result

Let's consider the case with 1+1 dimensions instead of 3+1:

$$E(d) \longmapsto E(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n$$

By introducing a we get:

$$\begin{aligned} E(d) &= \frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-an\pi/d} = \\ &= -\frac{1}{2} \frac{\partial}{\partial a} \sum_{n=1}^{\infty} e^{-an\pi/d} = \\ &= -\frac{1}{2} \frac{\partial}{\partial a} \frac{1}{1-e^{-a\pi/d}} = \\ &= -\frac{1}{2} \frac{\partial z}{\partial a} \frac{\partial}{\partial z} \frac{1}{1-z} = \\ &= -\frac{1}{2}\left(-\frac{\pi}{d} z\right)\left(-\frac{1}{(1-z)^2}\right) = \\ &= -\frac{\pi}{2d} \frac{z}{(1-z)^2} \qquad \text{where } z = e^{-a\pi/d} \end{aligned}$$

As $a/d \ll 1$, $z \sim 1$

However $f(z) = \frac{z}{(1-z)^2}$ has a pole of order 2

We thus need to expand using a Laurent series: $f(z) = \sum_{n=-\infty}^{\infty} a_j (z-1)^j$ where $a_j = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-1)^{j+1}} dz$ $\qquad \gamma: |z-1| = 1$

Particles

As $[H, a^\dagger_{\vec{p}}] = \omega_p a^\dagger_p$ and $[H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}}$ we have that $|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle$ and $H|\vec{p}\rangle = \omega_{\vec{p}}|\vec{p}\rangle$

We can interpret $|\vec{p}\rangle$ as the momentum eigenstate of a single particle of mass m as $E^2 = \vec{p}^2 + m^2$ (i.e. Relativistic energy)

$\hookrightarrow$ particles are created by disturbing the vacuum. This effect causes the application of $a^\dagger_{\vec{p}}$ $\longrightarrow$ type of particle depends on $a^\dagger_{\vec{p}}$ and thus fields

The momentum $\vec{P}$ (See E-M Tensor) can be turned into an operator as follows:

$$\Pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left[a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$\vec{\nabla}\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} i(\sqrt{2\omega_p})^{-1}\left[a_{\vec{p}}\vec{p}e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\vec{p}e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$\Pi(\vec{x})\vec{\nabla}\phi(y) = \int \frac{d^3p\,d^3q}{(2\pi)^3}\frac{1}{2}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}\vec{q}\left[a_{\vec{p}}a_{\vec{q}}e^{i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} - \left(a_{\vec{p}}a^\dagger_{\vec{q}}e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + a^\dagger_{\vec{p}}a_{\vec{q}}e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})}\right)\right]$$

$$\vec{P} = -\int d^3x\,\Pi(x)\vec{\nabla}\phi(x) = -\frac{1}{2}\int\frac{d^3x\,d^3p\,d^3q}{(2\pi)^6}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}\vec{q}\left[a_{\vec{p}}a_{\vec{q}}e^{i(\vec{p}+\vec{q})\cdot\vec{x}} + a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}e^{-i(\vec{p}+\vec{q})\cdot\vec{x}} - \left(a_{\vec{p}}a^\dagger_{\vec{q}}e^{i(\vec{p}-\vec{q})\cdot\vec{x}} + a^\dagger_{\vec{p}}a_{\vec{q}}e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}\right)\right] =$$

$$= -\frac{1}{2}\int\frac{d^3p\,d^3q}{(2\pi)^3}\vec{q}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}\left[(a_{\vec{p}}a_{\vec{q}} + a^\dagger_{\vec{p}}a^\dagger_{\vec{q}})\delta(\vec{p}+\vec{q}) - (a_{\vec{p}}a^\dagger_{\vec{q}} + a^\dagger_{\vec{p}}a_{\vec{q}})\delta(\vec{p}-\vec{q})\right] =$$

$$= \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\vec{p}\left[a_{\vec{p}}a_{-\vec{p}} + a^\dagger_{\vec{p}}a^\dagger_{-\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}a_{\vec{p}}\right] =$$

$$= \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\vec{p}\left[\underline{a_{\vec{p}}a_{-\vec{p}} + a^\dagger_{\vec{p}}a^\dagger_{-\vec{p}}} + (2\pi)^3\delta(0) + 2a^\dagger_{\vec{p}}a_{\vec{p}}\right]$$

As $\vec{p}\,(a_{\vec{p}}a_{-\vec{p}} + a^\dagger_{\vec{p}}a^\dagger_{-\vec{p}})$ is antisymmetric w.r.t $\vec{p} \longleftrightarrow -\vec{p}$ (i.e. odd), $\int d^3p\,\vec{p}\,(a_{\vec{p}}a_{-\vec{p}} + a^\dagger_{\vec{p}}a^\dagger_{-\vec{p}}) = 0$

As a result: $\vec{P} = \frac{1}{2}\int d^3p\,\delta(0) + \int\frac{d^3p}{(2\pi)^3}\vec{p}\,a^\dagger_{\vec{p}}a_{-\vec{p}}$

Applying to $|0\rangle$: $\vec{P}|0\rangle = \left[\frac{1}{2}\int d^3p\,\delta(0)\right]|0\rangle$

Then, after normal ordering: $\vec{P} = \int\frac{d^3p}{(2\pi)^3}\vec{p}\,a^\dagger_{\vec{p}}a_{-\vec{p}}$

Silarly, we can get the angular momentum operator from the EM Tensor:

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$$

$$J^i = Q^{ij} = \int d^3x\,(x^i T^{0j} - x^j T^{0i}) = \int d^3x\,(\mathcal{J}^0)^{ij} = \varepsilon^{ijk}\int d^3x\,(\mathcal{J}^0)^{jk}$$

Applying the operator $\vec{P}$ on the single particle states $|\vec{p}\rangle$ we get $\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$ i.e. $|\vec{p}\rangle$ has momentum $\vec{p}$

Applying the operator J^i on $|\vec{p}=0\rangle$, $J^i|\vec{p}=0\rangle = 0$ i.e. Quantization of scalar field gives rise to particle with internal ang. mom (Spin) zero

Multi-Particle States

A multi particle state is a state created by the action of multiple $a^\dagger_{\vec{p}_i}$ i.e. n-particle state: $|\vec{p}_1, \dots, \vec{p}_n\rangle = a^\dagger_{\vec{p}_1}\dots a^\dagger_{\vec{p}_n}|0\rangle$

As $[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0\ \forall \vec{p},\vec{q} \in \mathbb{R}^3$, $|\vec{p},\vec{q}\rangle = |\vec{q},\vec{p}\rangle$ and the particles are thus (spin-0) bosons (Symmetric w.f. for $\vec{p}=\vec{q}$)

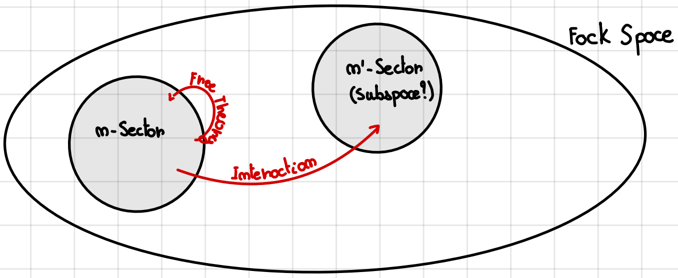
The Hilbert space related to a scalar field is known as "Fock space" and is spanned by all possible multi-particle states i.e. $|0\rangle$, $a^\dagger_{\vec{p}}|0\rangle$, $a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}|0\rangle$, ...

The Fock space can be viewed as the sum of all n-particle Hilbert spaces ($n \geq 0$) i.e. generalisation of Hilbert space to infinite particles

The number of particles in any given state is given by the number operator $N = \int\frac{d^3p}{(2\pi)^3}a^\dagger_{\vec{p}}a_{\vec{p}}$ which satisfies $N|\vec{p}_1, \dots, \vec{p}_n\rangle = n|\vec{p}_1, \dots, \vec{p}_n\rangle$

The number operator commutes with (free theories) hamiltonian i.e. $[N, H] = 0$ and in free theories particle number is conserved as there are no potentials/interactions

On the other hand, once interactions are introduced particles can be created/destroyed



Operator Valued Distribution

The particle states $|\vec{p}\rangle$ are momentum eigenstates but not position eigenstates and thus not localized → remember Heisenberg's Uncertainty Principle

This due to the fact that no momentum or position eigenstate can be normalized i.e. $\langle 0|a_{\vec{p}}a^\dagger_{\vec{p}}|0\rangle = \langle \vec{p}|\vec{p}\rangle = (2\pi)^3\delta(0)$ & $\langle 0|\phi(\vec{x})\phi(\vec{x})|0\rangle = \langle \vec{x}|\vec{x}\rangle = \delta(0)$
As such, $a_{\vec{p}}$ and $\phi(\vec{x})$ are not good operators on the Fock space.

We can construct good, normalizable states by considering the superposition of multiple $|\vec{p}\rangle$ states i.e. the construction of a wavepocket $|\psi(\vec{x})\rangle$
Viewed from the point of view of $|\psi(\vec{x})\rangle$ this is its Fourier decomposition in constituting $|\vec{p}\rangle$ states

$$|\psi(x)\rangle = \int \frac{d^3p}{(2\pi)^3}\varphi(\vec{p})|\vec{p}\rangle e^{-i\vec{p}\cdot\vec{x}}$$ where $\varphi(\vec{p})$ is responsible for the normalization e.g. $\varphi(\vec{p}) = e^{-\vec{p}^2/2m^2}$ s.t. $\int \frac{d^3p}{(2\pi)^3}|\varphi(\vec{p})|^2 = 1$

$$\langle\psi(x)|\psi(x)\rangle = \int \frac{d^3p\,d^3q}{(2\pi)^6}\varphi^\dagger(\vec{q})\varphi(\vec{p})\langle\vec{q}|\vec{p}\rangle e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} = \int\frac{d^3p\,d^3q}{(2\pi)^6}\varphi^\dagger(\vec{q})\varphi(\vec{p})\left[(2\pi)^3\delta(\vec{p}-\vec{q})\langle 0|0\rangle\right]e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} = \int\frac{d^3p}{(2\pi)^3}|\varphi(\vec{p})|^2\langle 0|0\rangle =$$

Relativistic Normalization

From the vacuum state $|0\rangle$ we construct the single particle states $|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle$
As $|0\rangle$ must be normalized we have: $\langle 0|0\rangle = 1$ and $\langle\vec{p}|\vec{q}\rangle = (2\pi)^3\delta(\vec{p}-\vec{q})$ which is ∞ when $\vec{p}=\vec{q}$ but zero otherwise
Why $\langle\vec{p}|\vec{q}\rangle = (2\pi)^3\delta(\vec{p}-\vec{q})$?

$$\langle\vec{p}|\vec{q}\rangle = \langle a^\dagger_{\vec{p}}|0\rangle\,|\,a^\dagger_{\vec{q}}|0\rangle\rangle = \langle 0|a_{\vec{p}}a^\dagger_{\vec{q}}|0\rangle = \langle 0|a^\dagger_{\vec{q}}\underbrace{a_{\vec{p}}|0\rangle}_{0} + [a_{\vec{p}},a^\dagger_{\vec{q}}]\langle 0|0\rangle = (2\pi)^3\delta(\vec{p}-\vec{q})$$

Are these normalization relationships invariant?

Momentum Lorentz Transform: $p^\mu \longmapsto (p')^\mu = \Lambda^\mu{}_\nu p^\nu$ s.t. $|\vec{p}\rangle \longmapsto |\vec{p}'\rangle$
Ideally, we have: $|\vec{p}\rangle \longmapsto |\vec{p}'\rangle = U(\Lambda)|\vec{p}\rangle$ s.t. $\langle\vec{p}'|\vec{p}'\rangle = \langle U(\Lambda)\vec{p}|U(\Lambda)\vec{p}\rangle = \langle\vec{p}|U^\dagger U|\vec{p}\rangle = \langle\vec{p}|\vec{p}\rangle$
However, if $\vec{p}'$ is not norm. : $|\vec{p}\rangle \longmapsto \lambda(\vec{p},\vec{p}')|\vec{p}'\rangle$ i.e. $\langle\lambda\vec{p}'|\lambda\vec{p}'\rangle$ needs to be equal to $\langle\vec{p}|\vec{p}\rangle$

We want our momentum state to be normalized in any frame i.e. we want $\langle\vec{p}|\vec{q}\rangle$ to be Lorentz invariant
However, $\vec{p}$ and $\vec{q}$ are 3-vectors and in general $\delta(\vec{p}-\vec{q}) \neq \delta(\vec{p}'-\vec{q}')$ where $\vec{p}', \vec{q}'$ are the transformed 3-vectors
To find a normalization that is frame invariant we consider the (identity) Projection operator:

Scalar (invariant) quantity: $1 = \int\frac{d^3p}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}| \Longrightarrow |\vec{q}\rangle = \int\frac{d^3p}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}|\vec{q}\rangle$

While it is invariant as a whole, d^3p and $|\vec{p}\rangle\langle\vec{p}|$ are not while d^4p and $\delta(0)$ are
Thus the combination $d^4p\,\delta^{(4)}(0)$ must be Lorentz invariant. It follows that:

Lorentz Inv. Int: $$\int\frac{d^4p}{(2\pi)^3}\delta(0) = \int\frac{d^4p}{(2\pi)^3}\delta(p_\mu p^\mu - m^2) = \int\frac{dp^3}{(2\pi)^3}dp_0\,\delta(p_0^2-\vec{p}^2-m^2)\Big|_{p_0>0} =$$
$$= \int\frac{d^3p}{(2\pi)^3}dp_0\,\delta(p_0^2-E_{\vec{p}}^2)\Big|_{p_0>0} = \int\frac{d^3p}{(2\pi)^3}\frac{dp_0}{2p_0}\left[\delta(p_0-E_{\vec{p}})+\delta(p_0+E_{\vec{p}})\right]\Big|_{p_0>0} = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\ \Big]\ \text{Lorentz Invariant}$$

Note: As $\int d^3p\,\delta^{(3)}(\vec{p}-\vec{q}) = 1$ we have that $\int\frac{d^3p}{2E_{\vec{p}}}2E_{\vec{p}}\delta^{(3)}(\vec{p}-\vec{q})$ is also Lorentz invariant

Consequence: The Lorentz Invariant Dirac Delta function is $2E_{\vec{p}}\delta(\vec{p}-\vec{q})$ such that the relativistically invariant normalization is given by $\langle p|q\rangle = (2\pi)^3(2E_{\vec{p}})\delta^{(3)}(\vec{p}-\vec{q})$ where the relativistically normalized state are $|p\rangle = \sqrt{2E_{\vec{p}}}|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\,a^\dagger_{\vec{p}}|0\rangle$

Then: $$1 = \int\frac{d^3p}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}| = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}|p\rangle\langle p|$$

Complex Scalar Field

Consider a complex scalar field $\varphi(x)$ with Lagrangian density $\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - M^2 \varphi^* \varphi$

A complex scalar field can be written as a linear superposition of two real scalar fields ϕ_1, ϕ_2:

- $\varphi(\vec{x}) = [\phi_1(\vec{x}) + i\,\phi_2(\vec{x})]/\sqrt{2}$
- $\varphi^*(\vec{x}) = [\phi_1(\vec{x}) - i\,\phi_2(\vec{x})]/\sqrt{2}$

As two ϕ_i, two equations of motion
$\hookrightarrow$ Treat $\varphi(\vec{x}), \varphi^*(\vec{x})$ separately

The equations of motion are:

- $\partial_\mu \partial^\mu \varphi + M^2 \varphi = 0$
- $\partial_\mu \partial^\mu \varphi^* + M^2 \varphi^* = 0$

These result into the following definitions of the fields and conjugate momentum:

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(b_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} + c^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right) \qquad \pi = \int \frac{d^3p}{(2\pi)^3}\, i \sqrt{\frac{E_{\vec{p}}}{2}} \left(b^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} - c_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}}\right) = \dot{\varphi}^*$$

$$\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(b^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}}\right) \qquad \pi^\dagger = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(b_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} - c^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right) = \dot{\varphi}$$

N.B. As φ and φ^* are not real, the fields and momentum are not hermitian i.e. $b \neq c$

Commutation relationship

$$[\psi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \quad \text{and} \quad [\psi(\vec{x}), \pi^\dagger(\vec{y})] = 0$$

$$[\varphi(\vec{x}), \varphi(\vec{y})] = [\varphi(\vec{x}), \varphi^\dagger(\vec{y})] = 0$$

$$[b_{\vec{p}}, b^\dagger_{\vec{q}}] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})$$
$$[c_{\vec{p}}, c^\dagger_{\vec{q}}] = (2\pi)^3\, \delta^{(3)}(\vec{p}-\vec{q})$$
$$[b_{\vec{p}}, b_{\vec{q}}] = [c_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c^\dagger_{\vec{q}}] = 0$$

Consequences

The quantisation of complex scalar field gives rise to two creation operators $b^\dagger, c^\dagger$, one for every scalar field. Each of these operators corresponds to the creation of two types of particles, both with mass M and spin-0. However, these particles correspond to different fields and thus have diff. quantum number. These two particles are labeled as particle and antiparticles.

The conserved charge: $Q = i\int d^3x\, (\dot{\varphi}^* \varphi - \varphi^* \dot{\varphi}) = i\int d^3x\, (\pi\varphi - \varphi^* \pi^\dagger) = \int \frac{d^3p}{(2\pi)^3} (c^\dagger_{\vec{p}} c_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}}) = N_c - N_b \Longrightarrow$ Comes from Internal Symmetry

$N_c \equiv$ Number of particles created by $c^\dagger$

$N_b \equiv$ Number of anti-particles created by $b^\dagger$

In free theories N_c, N_b are separately conserved but in interaction they are not. Nonetheless, in both cases Q is conserved i.e. $[H, Q]$

Heisemberg Picture

In Schrödinger's picture, operators such as $\phi(\vec{x})$ and $\Pi(\vec{x})$ are not time dependent but the states $|\vec{p}(t)\rangle = e^{-iE_{\vec{p}}t}|\vec{p}\rangle$ are.
It is does not evident that results derived from the Lorentz invariant remain invariant after quantisation
However, the Heisemberg picture makes Lorentz Invariance more manifest

Heisenberg Picture

In Heisenberg's Picture, time dependence is assigned to operators and not to the states

An operator O can be defined Heisenberg's picture (i.e. O_H) in terms of the Schrödinger's picture operator O_S as follows: $O_H = e^{iHt}O_S e^{-iHt}$

$\hookrightarrow$ $O_H = e^{iHt}O_S e^{-iHt}$

$$\dot{O}_H = \left[\left(\frac{\partial}{\partial t}+\frac{\partial H}{\partial t}\frac{\partial}{\partial H}\right)e^{iHt}\right]O_S e^{-iHt} + e^{iHt}\left[\left(\frac{\partial}{\partial t}+\frac{\partial H}{\partial t}\frac{\partial}{\partial H}\right)O_S\right]e^{-iHt} + e^{iHt}O_S\left[\left(\frac{\partial}{\partial t}+\frac{\partial H}{\partial t}\frac{\partial}{\partial H}\right)e^{-iHt}\right] =$$

$$= i\left(H+\frac{\partial H}{\partial t}t\right)O_H + e^{iHt}\frac{\partial O_S}{\partial t}e^{-iHt} - iO_H\left(H+\frac{\partial H}{\partial t}t\right) = i[H,O_H] + e^{iHt}\frac{\partial O_S}{\partial t}e^{-iHt}$$

In QFT, we lift the subscripts S and H in favor of labeling operators in Schrödinger's picture by $\vec{x}$ (Position 3-vector) and operators in Heisenberg's picture by $x^\mu = (\vec{x},t)$ (or simply x) i.e. spacetime position. It follows that:

Schrödinger: $\phi(\vec{x}) \longmapsto$ Heisenberg: $\phi(x) = \phi(\vec{x},t) = e^{iHt}\phi(\vec{x})e^{-iHt}$ s.t. $\phi(\vec{x},0) = \phi(\vec{x})$

Schrödinger: $\Pi(\vec{x}) \longmapsto$ Heisenberg: $\Pi(x) = \Pi(\vec{x},t) = e^{iHt}\Pi(\vec{x})e^{-iHt}$ s.t. $\Pi(\vec{x},0) = \Pi(\vec{x})$

Commutation Relations

$$[O_H^{(1)}(t_1), O_H^{(2)}(t_2)] = e^{iHt_1}O_S^{(1)}e^{-iHt_1}e^{iHt_2}O_S^{(2)}e^{-iHt_2} - e^{iHt_2}O_S^{(2)}e^{-iHt_2}e^{iHt_1}O_S^{(1)}e^{-iHt_1} = e^{iHt_1}O_S^{(1)}e^{iH(t_2-t_1)}O_S^{(2)}e^{-iHt_2} - e^{iHt_2}O_S^{(2)}e^{iH(t_1-t_2)}O_S^{(1)}e^{-iHt_1}$$

If $t_1=t_2$: $[O_H^{(1)}(t), O_S^{(2)}(t)] = e^{iHt}[O_S^{(1)}, O_S^{(2)}]e^{-iHt} = [O_S^{(1)}, O_S^{(2)}]$ $\Longrightarrow$ Commutator Relations at equivalent times are the same as in Sch. Picture

Theo: $[\phi_a(\vec{x},t), \phi_b(\vec{y},t)] = [\Pi_a(\vec{x},t), \Pi_b(\vec{y},t)] = 0$ $\quad [\phi_a(\vec{x},t), \Pi_b(\vec{y},t)] = i\delta(\vec{x}-\vec{y})\delta^a_b$

Evolution of the fields

Consider the scalar field ϕ and the related Hamiltonian $H = \frac{1}{2}\int d^3x\, \left[\Pi^2(\vec{x}) + (\vec{\nabla}\phi(\vec{x}))^2 + m^2\phi^2(\vec{x})\right]$
For such a scalar field we know that the it must satisfy $\partial_\mu\partial^\mu\phi + m^2\phi = 0$

As $\phi(x) = \phi(\vec{x},t)$ we can now study the time evolution of the fields
By $\dot{O}_H = i[H,O_H]$ we know that:

$$\dot{\phi} = i[H,\phi] = iH\phi(\vec{x},t) - i\phi(\vec{x},t)H = \frac{i}{2}\int d^3y\left\{\left[\Pi^2(y) + (\vec{\nabla}\phi(y))^2 + m^2\phi^2(y)\right]\phi(x) - \phi(x)\left[\Pi^2(y) + (\vec{\nabla}\phi(y))^2 + m^2\phi^2(y)\right]\right\} =$$

$$= \frac{i}{2}\int d^3y\left\{[\Pi^2(y), \phi(x)] + [(\vec{\nabla}\phi(y))^2, \phi(x)] + m^2[\phi^2(y), \phi(x)]\right\}$$

$$\dot{\Pi} = i[H,\Pi] = iH\Pi(\vec{x},t) - i\Pi(\vec{x},t)H = \frac{i}{2}\int d^3y\left\{\left[\Pi^2(y) + (\vec{\nabla}\phi(y))^2 + m^2\phi^2(y)\right]\Pi(x) - \Pi(x)\left[\Pi^2(y) + (\vec{\nabla}\phi(y))^2 + m^2\phi^2(y)\right]\right\} =$$

$$= \frac{i}{2}\int d^3y\left\{[\Pi^2(y), \Pi(x)] + [(\vec{\nabla}\phi(y))^2, \Pi(x)] + m^2[\phi^2(y), \Pi(x)]\right\}$$

Commutators Relations

$[AB, C] = A[B,C] + [A,C]B \quad [A^2, B] = A[A,B] + [A,B]A$

$[(AB)^2, C] = (AB)[AB,C] + [AB,C](AB) = (AB)\left[A[B,C]+[A,C]B\right] + \left[A[B,C]+[A,C]B\right](AB)$

$$[\vec{\nabla}_y\phi(y), \phi(x)] = (\vec{\nabla}_y\phi(y))\phi(x) - \phi(x)(\vec{\nabla}_y\phi_y) = \vec{\nabla}_y(\phi(y)\phi(x) - \phi(x)\phi(y)) = \vec{\nabla}_y[\phi(y),\phi(x)] = 0$$

$[\Pi^2(y), \phi(x)] = \Pi(y)[\Pi(y),\phi(x)] + [\Pi(y),\phi(x)]\Pi(y) = -2i\delta(\vec{x}-\vec{y})\Pi(y)$

$[\phi^2(y), \phi(x)] = \phi(y)[\phi(y),\phi(x)] + [\phi(y),\phi(x)]\phi(y) = 0$

$[\Pi^2(y), \Pi(x)] = \Pi(y)[\Pi(y),\Pi(x)] + [\Pi(y),\Pi(x)]\Pi(y) = 0$

$[\phi^2(y), \Pi(x)] = \phi(y)[\phi(y),\Pi(x)] + [\phi(y),\Pi(x)]\phi(y) = 2i\delta(\vec{x}-\vec{y})\phi(y)$

$[\vec{\nabla}_y\phi(y), \phi(x)] = \vec{\nabla}_y[\phi(y),\phi(x)] = 0$

$[(\vec{\nabla}\phi(y))^2, \phi(x)] = (\vec{\nabla}\phi(y))[\vec{\nabla}\phi(y), \phi(x)] + [(\vec{\nabla}\phi(y)), \phi(x)](\vec{\nabla}\phi(y)) = 0$

$[\vec{\nabla}_y\phi(y), \Pi(x)] = \vec{\nabla}_y(\phi(y)\Pi(x) - \Pi(x)\phi(y)) = \vec{\nabla}_y[\phi(y),\Pi(x)] = i\vec{\nabla}_y\delta(\vec{x}-\vec{y})$

$[(\vec{\nabla}_y\phi(y))^2, \Pi(x)] = (\vec{\nabla}_y\phi(y))[\vec{\nabla}_y\phi(y), \Pi(x)] + [\vec{\nabla}_y\phi(y), \Pi(x)](\vec{\nabla}_y\phi(y)) = i\left[(\vec{\nabla}_y\phi(y))(\vec{\nabla}_y\delta(\vec{x}-\vec{y})) + (\vec{\nabla}_y\delta(\vec{x}-\vec{y}))(\vec{\nabla}_y\phi(y))\right]$

It follows that:

$$\dot{\phi} = i[H,\phi] = iH\phi(\vec{x},t) - i\phi(\vec{x},t)H = \frac{i}{2}\int d^3y \left\{\left[\Pi^2(y) + (\vec{\nabla}\phi(y))^2 + m^2\phi^2(y)\right]\phi(x) - \phi(x)\left[\Pi^2(y) + (\vec{\nabla}\phi(y))^2 + m^2\phi^2(y)\right]\right\} =$$

$$= \frac{i}{2}\int d^3y \left\{\left[\Pi^2(y), \phi(x)\right] + \left[(\vec{\nabla}\phi(y))^2, \phi(x)\right] + m^2\left[\phi^2(y), \phi(x)\right]\right\} =$$

$$= \int d^3y\, \delta(\vec{x}-\vec{y})\, \Pi(y) = \Pi(x)$$

$$\dot{\Pi} = i[H,\Pi] = iH\Pi(\vec{x},t) - i\Pi(\vec{x},t)H = \frac{i}{2}\int d^3y \left\{\left[\Pi^2(y) + (\vec{\nabla}\phi(y))^2 + m^2\phi^2(y)\right]\Pi(x) - \Pi(x)\left[\Pi^2(y) + (\vec{\nabla}\phi(y))^2 + m^2\phi^2(y)\right]\right\} =$$

$$= \frac{i}{2}\int d^3y \left\{\left[\Pi^2(y), \Pi(x)\right] + \left[(\vec{\nabla}\phi(y))^2, \Pi(x)\right] + m^2\left[\phi^2(y), \Pi(x)\right]\right\} =$$

$$= -\int d^3y \left\{\left[\vec{\nabla}_y \delta(\vec{x}-\vec{y})\right]\vec{\nabla}_y\phi(y) - m^2\phi(y)\,\delta(\vec{x}-\vec{y})\right\} =$$

$$= \nabla^2\phi(x) - m^2\phi = \ddot{\phi}$$

This proves that $\Pi(x) = \dot{\phi}(x)$ and that $\ddot{\phi}(x) - \nabla^2\phi + m^2\phi = \partial_\mu\partial^\mu\phi + m^2\phi = 0$

Fourier expansion of the field

We know that: $[H, a_{\vec{p}}] = -E_{\vec{p}}\, a_{\vec{p}}$ and $[H, a^\dagger_{\vec{p}}] = E_{\vec{p}}\, a^\dagger_{\vec{p}}$

The operators in the Heisenberg Picture are given by:

$$e^{iHt} a_{\vec{p}}\, e^{-iHt} = e^{-iE_{\vec{p}}t} a_{\vec{p}}$$
$$e^{iHt} a^\dagger_{\vec{p}}\, e^{-iHt} = e^{iE_{\vec{p}}t} a^\dagger_{\vec{p}}$$

Mapping $a_{\vec{p}} \longrightarrow e^{-iE_{\vec{p}}t} a_{\vec{p}}$ and $a^\dagger_{\vec{p}} \longrightarrow e^{iE_{\vec{p}}t} a^\dagger_{\vec{p}}$ in $\phi(\vec{x})$ we get $\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ipx} + a^\dagger_{\vec{p}}\, e^{ipx}\right)$ where $px = p_\mu x^\mu$

N.B

In Heisenberg's picture, operators such as $a_{\vec{p}}$, $a^\dagger_{\vec{p}}$, $\phi(x)$, ... have time dependence $\Longrightarrow$ Time evolution of $|\vec{p}\rangle$ and $|\vec{x}\rangle$ states is hidden within the operators

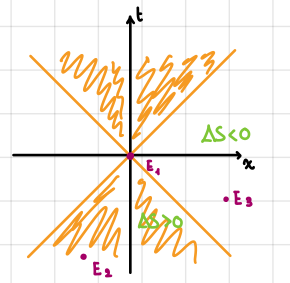
Therefore, the final states still evolve with time thanks to operators

Time evolution must be unitary (i.e. $|\varphi\rangle_H = U(t,t_0)|\varphi(t_0)\rangle$ s.t. $U^\dagger(t,t_0)U(t,t_0) = \mathbb{1}$?) such that total probability is conserved

Causality in Heisenberg's Picture

While the field $\phi(x)$ satisfies the Klein-Gordon equation, there is still some aspects of no Lorentz invariance
In fact the fields satisfy equal time commutation relations, we have no idea about arbitrary spacetime separations

E_1 and E_2 are causally connected
E_1 and E_2 are not causally connected



In order for our theory to be consistent with special relativity it needs to be causal
We thus want two operators to commute when applied to two events not causally connectes as one event should not affect the other
Two events x^μ and y^μ are not causally connected if and only if the space-time interval $s^2 = (x-y)^2 < 0$

We thus want our operators to satisfy the following: $[O_1(x), O_2(y)] = 0 \quad \forall (x-y)^2 < 0$
As our theory must satisfy this, let's check it by computing $[\phi(x), \phi(y)]$

$$\Delta(x-y) = [\phi(x), \phi(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \left\{ (a_{\vec p} e^{-ipx} + a^\dagger_{\vec p} e^{ipx})(a_{\vec q} e^{-iqy} + a^\dagger_{\vec q} e^{iqy}) - (a_{\vec q} e^{-iqy} + a^\dagger_{\vec q} e^{iqy})(a_{\vec p} e^{-ipx} + a^\dagger_{\vec p} e^{ipx}) \right\} =$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \left\{ a_{\vec p} a_{\vec q} e^{-i(px+qy)} + a_{\vec p} a^\dagger_{\vec q} e^{i(qy-px)} + a^\dagger_{\vec p} a_{\vec q} e^{i(px-qy)} + a^\dagger_{\vec p} a^\dagger_{\vec q} e^{i(px+qy)} - a_{\vec q} a_{\vec p} e^{-i(px+qy)} - a_{\vec q} a^\dagger_{\vec p} e^{i(px-qy)} - a^\dagger_{\vec q} a_{\vec p} e^{i(qy-px)} - a^\dagger_{\vec q} a^\dagger_{\vec p} e^{i(px+qy)} \right\}$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \left\{ [a_{\vec p}, a_{\vec q}] e^{-i(px+qy)} + [a_{\vec p}, a^\dagger_{\vec q}] e^{i(qy-px)} - [a_{\vec q}, a^\dagger_{\vec p}] e^{i(px-qy)} + [a^\dagger_{\vec p}, a^\dagger_{\vec q}] e^{i(px+qy)} \right\} =$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \left[e^{i(qy-px)} - e^{i(px-qy)} \right] \delta(\vec p - \vec q) =$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]$$

What are the features of $\Delta(x-y)$?

1) It is Lorentz invariant as there is $p(x-y)$ and the invariant measure $\int \frac{d^3p}{2E_{\vec p}}$
2) Does not vanish for causally connected events

   e.g. $x^\mu = (t_1, 0, 0, 0) \quad y^\mu = (t_2, 0, 0, 0) \quad$ s.t. $(x-y) = (t, 0, 0, 0)$

   $\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} (e^{-iE_{\vec p} t} - e^{iE_{\vec p} t})$
3) Vanishes for all $(x-y)^2 < 0$ Why?

Our theory is indeed Causal

Propagators

Some times we are iterested in determining the propality of finding at location x^μ a particle produced at y^μ
These probability is known as the propagator $D(x-y)$. It can be computed as follows:

$$\phi(x)\phi(y) = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \left\{ a_{\vec p} a_{\vec q} e^{-i(px+qy)} + a_{\vec p} a^\dagger_{\vec q} e^{i(qy-px)} + a^\dagger_{\vec p} a_{\vec q} e^{i(px-qy)} + a^\dagger_{\vec p} a^\dagger_{\vec q} e^{i(px+qy)} \right\} =$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \left\{ a_{\vec p} a_{\vec q} e^{-i(px+qy)} + a^\dagger_{\vec p} a_{\vec q} e^{i(px-qy)} + a^\dagger_{\vec q} a_{\vec p} e^{-i(px-qy)} + [a_{\vec p}, a^\dagger_{\vec q}] e^{-i(px-qy)} + a^\dagger_{\vec p} a^\dagger_{\vec q} e^{i(px+qy)} \right\} =$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip(x-y)} + \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \left\{ a_{\vec p} a_{\vec q} e^{-i(px+qy)} + a^\dagger_{\vec p} a_{\vec q} e^{i(px-qy)} + a^\dagger_{\vec q} a_{\vec p} e^{-i(px-qy)} + a^\dagger_{\vec p} a^\dagger_{\vec q} e^{i(px+qy)} \right\}$$

The propagator is therefore:

$$D(x-y) = \langle 0|\phi^\dagger(x)\phi(y)|0\rangle = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip(x-y)} \langle 0|0\rangle + \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \langle 0|a^\dagger_{\vec p} a^\dagger_{\vec q}|0\rangle e^{i(px+qy)} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip(x-y)}$$

Consequences of propagator description:

The propagator $D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip(x-y)}$ represents the probability of a particle produced at y^μ to be found at x^μ. Similarly, $D(y-x)$ is the probability of a particle produced at x^μ to be found at y^μ. If separation is spacelike i.e. $(x-y)^2 < 0$, $D(x-y) \sim e^{-m|\vec x - \vec y|}$ which means that the probability is exponentially decreasing but non-vanishing. How is this possible within a causal theory? While the propagator is non-vanishing outside the light cone, the commutator $[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0$. This can be interpreted as the non-zero amplitude of the particle travelling from $y \longrightarrow x$ cancelling the amplitude of the particle going $x \longrightarrow y$, leading to a net zero effect

Similarly, for a complex field: $[\psi(x), \psi^\dagger(y)] = 0$ and the particle $x \longmapsto y$ cancels antiparticle going $y \longmapsto x$

The Feynman Propagator

Time ordering: Symbolized by T, refers to ordering quantities by placing all operators evaluated at later times to the left e.g. $T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & y^0 > x^0 \end{cases}$

Feynman Propagator:

$$\Delta_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & y^0 > x^0 \end{cases}$$

Claim: Feynman Propagator can be written as: $\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)}$

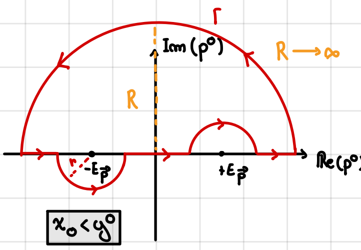
Proof:

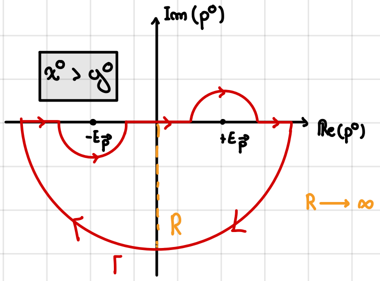
We have to show that, by integrating over p^0, we recover $D(x-y)$, $D(y-x)$

As $p^2 - m^2 = (p^0)^2 - \vec{p}^2 - m^2 = (p^0)^2 - (E_{\vec{p}})^2 = (p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})$, the integrand has 1st order pole at $p^0 = \pm E_{\vec{p}}$

In addition, for $f(p) = (p^2 - m^2)^{-1} e^{-ip(x-y)}$ we have: $f(p) = (p^0 - E_{\vec{p}})^{-1}(p^0 + E_{\vec{p}})^{-1} e^{-ip^0(x^0 - y^0) + i\vec{p}\cdot(\vec{x} - \vec{y})}$

- $\lim_{p^0 \to i\infty} f(p) = 0$ if $x^0 < y^0$
- $\lim_{p^0 \to -i\infty} f(p) = 0$ if $x^0 > y^0$





We can then apply the residue theorem: $\oint_\Gamma f(z)dz = \pm 2\pi i \sum_{k=1}^{N} \text{Res}(f, z_k)$ + if counter clockwise, - for counter clockwise

The $\text{Res}(f, z_k) = \frac{1}{(m-1)!} \lim_{z \to z_k} \frac{d^{m-1}}{dz^{m-1}}((z - z_k)^m f(z))$ where m is the order of the pole

It follows that:

- $\text{Res}(f, +E_{\vec{p}}) = \lim_{z \to E_{\vec{p}}} ((z - E_{\vec{p}}) f(z)) = \lim_{p^0 \to E_{\vec{p}}} \frac{i}{(p^0 + E_{\vec{p}})} e^{-ip^0(x^0 - y^0) + i\vec{p}\cdot(\vec{x} - \vec{y})} = \frac{i}{2E_{\vec{p}}} e^{-iE_{\vec{p}}(x^0 - y^0) + i\vec{p}\cdot(\vec{x} - \vec{y})}$

- $\text{Res}(f, -E_{\vec{p}}) = \lim_{z \to -E_{\vec{p}}} ((z + E_{\vec{p}}) f(z)) = \lim_{p^0 \to -E_{\vec{p}}} \frac{i}{(p^0 - E_{\vec{p}})} e^{-ip^0(x^0 - y^0) + i\vec{p}\cdot(\vec{x} - \vec{y})} = -\frac{i}{2E_{\vec{p}}} e^{-iE_{\vec{p}}(y^0 - x^0) + i(-\vec{p})\cdot(\vec{y} - \vec{x})}$

Flip sign of $\vec{p}$

Does not matter as integration boundaries are symm

Then we have:

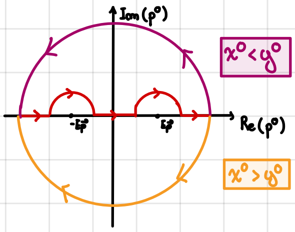
- $\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip(x-y)}$ with $p^0 = E_{\vec{p}}$ if $x^0 > y^0$
- $\Delta_F(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip(y-x)}$ with $p^0 = E_{\vec{p}}$ if $y^0 > x^0$

Green's Functions

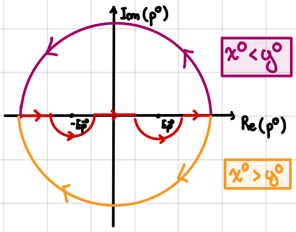
Applying the Klein-Gordon Equation to the Feynman Propagator we get:

$$(\partial_t^2 - \nabla^2 + m^2)\,\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, i\,\frac{m^2-p^2}{p^2-m^2}\, e^{-ip(x-y)} = -i\int \frac{d^4p}{(2\pi)^4}\, e^{-ip(x-y)} = -i\,\delta(x-y)$$ irrespective of contour

If we choose different contours:



Retarded Green Function



Advanced Green Function

Retarded Green Function: $\Delta_R(x-y) = \begin{cases} D(x-y) - D(y-x) & x^0 > y^0 \\ 0 & y^0 > x^0 \end{cases}$

Advanced Green Function: $\Delta_A(x-y) = \begin{cases} 0 & y^0 < x^0 \\ D(x-y) - D(y-x) & y^0 > x^0 \end{cases}$

$\Delta_R(x-y)$ and $\Delta_A(x-y)$ are used to solve the inhomogeneous KG equation $\partial_\mu\partial^\mu\phi + m^2\phi = \delta(x)$ is a source term

Δ_R is used if we know the initial field config. and we want to find what it evolves into

Δ_A is known if we know the end point of the field and we want to find where it came from

Non-Relativistic Fields

Consider the complex scalar fields $\varphi(\vec{x},t)$ and $\varphi^*(\vec{x},t)$ with Lagrangian density $\mathcal{L} = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi$

These satisfy the Klein-Gordon equation:

- $\partial_\mu\partial^\mu\varphi + m^2\varphi = 0$
- $\partial_\mu\partial^\mu\varphi^* + m^2\varphi^* = 0$

We can decompose the field into: $\varphi(\vec{x},t) = e^{-imt}\tilde{\varphi}(\vec{x},t)$ and $\varphi^*(\vec{x},t) = e^{+imt}\tilde{\varphi}^*(\vec{x},t)$

The KG equation turns into: $\partial_t^2\varphi - \nabla^2\varphi + m^2\varphi = \partial_t(-ime^{-imt}\tilde{\varphi} + e^{-imt}\dot{\tilde{\varphi}}) - e^{-imt}\nabla^2\tilde{\varphi} + e^{-imt}m^2\tilde{\varphi} =$

$= -m^2e^{-imt}\tilde{\varphi} - ime^{-imt}\dot{\tilde{\varphi}} - ime^{-imt}\dot{\tilde{\varphi}} + e^{-imt}\ddot{\tilde{\varphi}} - e^{-imt}\nabla^2\tilde{\varphi} + e^{-imt}m^2\tilde{\varphi} =$

$= e^{-imt}(\ddot{\tilde{\varphi}} - 2im\dot{\tilde{\varphi}} - \nabla^2\tilde{\varphi}) = 0$

Apply to the Lagrangian Density: $\mathcal{L} = \partial_\mu\varphi\,\partial^\mu\varphi^* - m^2\varphi^*\varphi = \dot{\varphi}^*\dot{\varphi} - \vec{\nabla}\varphi^*\vec{\nabla}\varphi - m^2\varphi^*\varphi =$

$= (ime^{imt}\tilde{\varphi}^* + e^{imt}\dot{\tilde{\varphi}}^*)(-ime^{-imt}\tilde{\varphi} + e^{-imt}\dot{\tilde{\varphi}}) - \vec{\nabla}\tilde{\varphi}^*\vec{\nabla}\tilde{\varphi} - m^2\tilde{\varphi}^*\tilde{\varphi} =$

$= m^2\tilde{\varphi}^*\tilde{\varphi} + im\tilde{\varphi}^*\dot{\tilde{\varphi}} - im\dot{\tilde{\varphi}}^*\tilde{\varphi} + \dot{\tilde{\varphi}}^*\dot{\tilde{\varphi}} - \vec{\nabla}\tilde{\varphi}^*\vec{\nabla}\tilde{\varphi} - m^2\tilde{\varphi}^*\tilde{\varphi} =$

$= im(\tilde{\varphi}^*\dot{\tilde{\varphi}} - \dot{\tilde{\varphi}}^*\tilde{\varphi}) - \vec{\nabla}\tilde{\varphi}^*\vec{\nabla}\tilde{\varphi}$

Non-relativistic limit: $|\vec{p}| \ll m \longmapsto |\ddot{\tilde{\varphi}}| \ll m|\dot{\tilde{\varphi}}|$ s.t. $i\dot{\tilde{\varphi}} \simeq -\frac{1}{2m}\nabla^2\tilde{\varphi}$ Similar to Sch. Equation for a free particle of mass m but no probability interpretation

$|\vec{p}| \ll m \longmapsto \dot{\varphi} \ll m\varphi$ s.t. $\mathcal{L} \simeq i\tilde{\varphi}^*\dot{\tilde{\varphi}} - \frac{1}{2m}\vec{\nabla}\tilde{\varphi}^*\vec{\nabla}\varphi$

The first order lagrangian is symmetric w.r.t. internal transformations of the kind: $\varphi \longmapsto e^{i\alpha}\varphi$

The corresponding current is: $j^\mu = \left(-\varphi^*\varphi, \frac{i}{2m}(\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*)\right)$

What about the Hamiltonian:

Conjugate Momentum: $\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\tilde{\varphi}}} = i\tilde{\varphi}^* \Longrightarrow$ The conj. momenta of $\tilde{\varphi}$ is $i\tilde{\varphi}^*$ which makes sense as to specify a 1st order system it is enough to specify φ, φ^* at $t=0$

It follows that: $\mathcal{H} = \dot{\varphi}\Pi - \mathcal{L} = \frac{1}{2m}\vec{\nabla}\tilde{\varphi}^*\vec{\nabla}\tilde{\varphi}$

To quantize the Hamiltonian we impose the following relations: $[\varphi(\vec{x}),\varphi(\vec{y})] = [\varphi^\dagger(\vec{x}),\varphi^\dagger(\vec{y})] = 0$ and $[\varphi(\vec{x}),\varphi^\dagger(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})$ (Sch. Picture)

As this is a 1st order Lagrangian, φ has just one solution of the form $\varphi(\vec{x}) = Ae^{i\vec{p}\cdot\vec{x}}$

The Fourier expansion is $\varphi(\vec{x}) = \int\frac{d^3p}{(2\pi)^3}a_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}$ and commutation relations $[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta(\vec{p}-\vec{q})$

By Plugging in the Fourier Transform and using commutation relations we get: $H|\vec{p}\rangle = \frac{\vec{p}^{\,2}}{2m}|\vec{p}\rangle$

Thus, quantizing the 1st order lagrangian leads to:

- 1 single type of particle $\Longrightarrow$ Anti-particles are consequence of relativity
- The conserved charge $Q = \int d^3x\,\varphi^\dagger\varphi$ is the particle number and remains conserved for interactions
- No non-relativistic limit of real scalar field as particles are their own antiparticles

Recovering QM

In QM: $\vec{X}$ and $\vec{P}$ are operators

relativistic

In QFT: Only $\vec{P}$ is an operator ($\vec{x}$ is not talked about as single particle states are only localized in momentum space but not in position space)

In the non-relativistic limit:

Operator $\varphi^\dagger(\vec{x}) = \int\frac{d^3p}{(2\pi)^3}a^\dagger_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}}$ on vacuum $\varphi^\dagger(\vec{x})|0\rangle = \int\frac{d^3p}{(2\pi)^3}a^\dagger_{\vec{p}}|0\rangle e^{-i\vec{p}\cdot\vec{x}} = \int\frac{d^3p}{(2\pi)^3}|\vec{p}\rangle e^{-i\vec{p}\cdot\vec{x}}$

By wave packet interpretation this can be interpreted as a particle state localized at position $|\vec{x}\rangle$

It also follows that the position operator $\vec{X} = \int d^3x\,\vec{x}\,\varphi^\dagger(\vec{x})\varphi(\vec{x})$ s.t. $\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$

Let's now construct the Schrödinger's wavefunction by superimposing one particle states i.e. $|\varphi\rangle = \int d^3x\, \varphi(\vec{x})|\vec{x}\rangle$

It follows that: $X^i|\varphi\rangle = \int d^3x\, x^i \varphi(\vec{x})|\vec{x}\rangle$

What about momentum? The operator is $\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$

From which follows that $\vec{P}|\vec{x}\rangle = \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}} |\varphi\rangle$

$$= \int d^3x \frac{d^3p\, d^3q}{(2\pi)^6} \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}} a^\dagger_{\vec{q}} \varphi(\vec{x}) e^{-i\vec{q}\cdot\vec{x}}|0\rangle =$$

$$= \int d^3x \frac{d^3p\, d^3q}{(2\pi)^6} \vec{p} a^\dagger_{\vec{p}} \left[a^\dagger_{\vec{q}} a_{\vec{p}} + [a_{\vec{p}}, a^\dagger_{\vec{q}}]\right] \varphi(\vec{x}) e^{-i\vec{q}\cdot\vec{x}}|0\rangle =$$

$$= \int d^3x \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \varphi(\vec{x})|0\rangle =$$

$$= i\int \frac{d^3x\, d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} \vec{\nabla}(e^{-i\vec{p}\cdot\vec{x}})\varphi(\vec{x})|0\rangle =$$

$$= i\int \frac{d^3x\, d^3p}{(2\pi)^3} \left[\vec{\nabla}(e^{-i\vec{p}\cdot\vec{x}})\varphi(\vec{x})\right] a^\dagger_p|0\rangle =$$

$$= i\int \frac{d^3x\, d^3p}{(2\pi)^3} \left\{\vec{\nabla}\left[e^{-i\vec{p}\cdot\vec{x}}\varphi(\vec{x}) a^\dagger_{\vec{p}}|0\rangle\right] - \vec{\nabla}(\varphi(\vec{x}))\, e^{-i\vec{p}\cdot\vec{x}} a^\dagger_{\vec{p}}\right\}|0\rangle =$$

$$= -i\int \frac{d^3x\, d^3p}{(2\pi)^3} \vec{\nabla}(\varphi(\vec{x}))|\vec{p}\rangle e^{-i\vec{p}\cdot\vec{x}} = \int d^3x \left[-i\vec{\nabla}(\varphi(\vec{x}))\right]|\vec{x}\rangle$$

Therefore, the position and momentum operators act on single particle states just like they do in QM and have $[X^j, P^k] = i\delta^{jk}|\varphi\rangle$

In addition:

$H = \int d^3x \frac{1}{2m} \vec{\nabla}\tilde{\varphi}^* \vec{\nabla}\tilde{\varphi} = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}}$ by plugging φ^* and φ's Fourier Transforms

Then $H|\varphi\rangle = \int \frac{d^3x\, d^3p}{(2\pi)^3} \frac{\vec{p}^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}} \varphi(\vec{x})|\vec{x}\rangle =$ Similar to $\vec{P}$ derivation

$$= -\frac{1}{2m}\int d^3x \left(\nabla^2\varphi(\vec{x})\right)|\vec{x}\rangle$$

$i\frac{\partial|\varphi\rangle}{\partial t} =$

Summary and Consequences

The complex scalar field Lagrangian Density is given by: $\mathcal{L} = \partial_\mu\varphi^*\partial_\mu\varphi - m^2\varphi^*\varphi$

The field satisfy the KG equations: $\partial_\mu\partial^\mu\varphi + m^2\varphi = 0$ and $\partial_\mu\partial^\mu\varphi^* + m^2\varphi$

We are free to decompose fields however we want to e.g. $\varphi = e^{-imt}\tilde{\varphi}$, $\varphi^* = e^{imt}\tilde{\varphi}^*$

By applying these decompositions we can get KG equationd and Lagrangian in terms of $\tilde{\varphi}$ and $\tilde{\varphi}^*$

By the applying the non-relativistic limit i.e. $|\vec{p}| \ll m$ we can get the non relativistic equations:

$i\partial_t\tilde{\varphi} = -(2m)^{-1}\nabla^2\tilde{\varphi}$ if $|\dot{\tilde{\varphi}}| \ll m|\dot{\tilde{\varphi}}|$ $\Longrightarrow$ ~ Sch. equation for free particle, however no probabilistic interpretation

$\mathcal{L} \approx i\tilde{\varphi}^*\dot{\tilde{\varphi}} - \frac{1}{2m}\vec{\nabla}\tilde{\varphi}^*\vec{\nabla}\tilde{\varphi}$ if $\partial_t\varphi \ll m\varphi$

These are first order Lagrangian and differential equations

From this Lagrangian it follows that:

- $j^\mu = \left(-\varphi^*\varphi, \frac{i}{2m}(\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*)\right)$ due to internal symmetry $\Longrightarrow$ The conserved charge $Q = \int d^3x\, \varphi^*\varphi$ is the particle number and it is conserved for any interaction
- Conjugate mom.: $\pi = \partial\mathcal{L}/\partial\dot{\varphi} = i\varphi^*$ and $\mathcal{H} = (2m)^{-1}\vec{\nabla}\varphi^*\vec{\nabla}\varphi$
- $\varphi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}}$ with $[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3\delta(\vec{p}-\vec{q})$ $\Longrightarrow$ Single operator because of 1st order. Therefore single type of particle and no antiparticle (which is a relativistic concept). It also follows that there is no non-relativistic field for real scalar fields as in that case particle is its own antiparticle.
  - $\hookrightarrow$ $|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle$ and $a_{\vec{p}}|0\rangle = 0$
  - $\hookrightarrow$ $H|\vec{p}\rangle = (\vec{p}^2/2m)|\vec{p}\rangle$ as $H = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}}$

In non relativistic limit:

Position localised particle state $|\vec{x}\rangle$ created as wavepacket i.e. $|\vec{x}\rangle = \psi^\dagger(\vec{x})|0\rangle$ where $\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}$

As $|\vec{x}\rangle = \psi^\dagger(\vec{x})|0\rangle$ and $|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle$ we have

- Position operator: $\vec{X} = \int d^3x \; \vec{x} \, \psi^\dagger(\vec{x}) \, \psi(\vec{x})$
- Momentum operator: $\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \, a^\dagger_{\vec{p}} a_{\vec{p}}$

We can define the Sch. Wavefunction as $|\varphi\rangle = \int d^3x \; \varphi(\vec{x}) |\vec{x}\rangle$ where $\int |\varphi(x)|^2 d^3x = 1$

We can thus show that:

$$\left.\begin{aligned} X^i|\varphi\rangle &= \int d^3x \; (x^i)\,\varphi(\vec{x})\,|\vec{x}\rangle \\ P^i|\varphi\rangle &= \int d^3x \left(-i\frac{\partial}{\partial x^i}\varphi(\vec{x})\right)|\vec{x}\rangle \end{aligned}\right\} [x^j, p^k] = i\delta^{jk}$$

Quantum Mechanics!!

Similarly: $H|\varphi\rangle = -\frac{1}{2m}\int d^3x \left[\nabla^2\varphi(x)\right]|\vec{x}\rangle$

From which follows: $i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\nabla^2\varphi$ i.e. Sch. Equation but this time with probability interpretation

Why $Q = \int d^3x \; |\varphi(\vec{x})|^2$

Interacting Fields

Interactions

Often particles move in some fixed background potential $V(\vec{x})$

The addition to the Lagrangian Density is of the form $\Delta\mathcal{L} = -V(\vec{x})\psi^*\psi$

If we have a system of n-particles $(n \geq 2)$ we expect to have interactions between particles

The addition the lagrangian is of the form: $\Delta\mathcal{L} = \psi^*(\vec{x})\psi^*(\vec{x})\psi(\vec{x})\psi(\vec{x})$

This corresponds to the annihilation of two particles and the creation of two other particles

Small and Big Interactions

Not all interactions are always relevant: some are more important at low energy while others are more important at high energies

For example, consider the real scalar field Lagrangian Density: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n$

The λ_n parameters are called "coupling constant"

As $\mathcal{L}$ has units $[\mathcal{L}] = 4$ (i.e. energy^4) and $[\phi] = 1$ we have that $\boxed{[\lambda_n] = 4 - n}$

Clearly, the behaviour of each interaction scales differently with energy

We are interested in small perturbations:

If we define E as the energy scale of the interaction we get 3 types of interaction based on their coupling constants λ_n

- Relevant i.e. $[\lambda_3] = 1$

  These terms are dimensionless for λ_3/E which means that at energies $E \gg \lambda_3$ are very small perturbations (compared to other terms of the Lagrangian)

  while are big perturbations for $\lambda_3 \ll E$
- Marginal i.e. $[\lambda_4] = 0$

  These are dimensionless and thus small if $\lambda_4 \ll 1$
- Irrelevant i.e. $[\lambda_n] < 0$ if $n \geq 5$

  Dimensionless parameter is $\lambda_n E^{n-4}$, which is small at low energies and high at high energies

N.B. Suppose we find a TOE that describes everything at the energy scale Λ. However, we are interested in scale $E \ll \Lambda$

We can write $\boxed{\lambda_n = g_n \Lambda^{4-n}}$ where $g_n \sim \mathcal{O}(1)$. Therefore, as $\boxed{(E/\Lambda)^{n-4} \lll 1 \text{ for } n > 4, \text{ these are heavily suppressed}}$

Interaction Picture

Schrodinger Picture:

- States depend on time $i\frac{d|\psi\rangle_s}{dt} = H|\psi\rangle_s$
- Operators are time independent

Heisenberg's Picture:

- States are fixed $|\psi\rangle_H = e^{iHt}|\psi\rangle_s$ s.t.

  $i\frac{d|\psi\rangle_s}{dt} = He^{-iHt}|\psi\rangle_H + ie^{-iHt}\frac{d|\psi\rangle_H}{dt} = He^{-iHt}|\psi\rangle_H \implies i\frac{d|\psi\rangle_H}{dt} = 0$
- Operators are time-dependent $O_H(t) = e^{iHt}O_S e^{-iHt}$

$\Longrightarrow$

Interaction Picture: Hybrid of Schrödingers & Heisenberg's Pictures

$\hookrightarrow$ Split Hamiltonian into $H = H_0 + H_{int}$ where:

- H_0 is the "Free Hamiltonian" which governs evolution of operators
- H_{int} is the "Interaction Hamiltonian" which governs evolution of states

N.B. Splitting is arbitrary but it generally pays off to include only int. in H_{int}

Consequence of Interaction Picture

$H = H_0 + H_{int} \Longrightarrow i|\dot{\psi}\rangle_s = H_0|\psi\rangle_s + H_{int}|\psi\rangle_s$

If $|\psi\rangle_s = e^{-iH_0t}|\psi\rangle_I \Longrightarrow |\dot{\psi}\rangle_I = e^{iH_0t}H_{int}e^{-iH_0t}|\psi\rangle_I = H_I|\psi\rangle_I$

Therefore, in Interaction Picture, we have:

- Hamiltonian: $H = H_0 + H_{int}$
- States: $|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_s$
- Operators: $O_I(t) = e^{iH_0t}O_S e^{-iH_0t}$ s.t. $H_I = e^{iH_0t}H_{int}e^{-iH_0t}$
- Sch. Equation: $i|\dot{\psi}\rangle_I = H_I|\psi\rangle_I$

Time evolution of states in Interaction Picture

States evolve according to an operator $U(t,t_0)$ s.t. $|\psi(t)\rangle_I = U(t,t_0)|\psi(t_0)\rangle$

What are the properties of this operator?

1) As probability is conserved, $U(t,t_0)$ is unitary i.e. $U^\dagger(t,t_0)\,U(t,t_0) = 1$

   $\langle\psi(t)|\psi(t)\rangle = \langle\psi(t_0)|U^\dagger(t,t_0)\,U(t,t_0)|\psi(t_0)\rangle = \langle\psi(t_0)|\psi(t_0)\rangle = 1$ iff $U^\dagger(t,t_0)\,U(t,t_0) = 1$

2) Evolution $t_0 \longrightarrow t_1 \longrightarrow t$ must be equal to $t_0 \longrightarrow t$ i.e. $U(t,t_0) = U(t,t_1)\,U(t_1,t_0)$

   $|\psi(t)\rangle = U(t,t_1)|\psi(t_1)\rangle = U(t,t_1)\,U(t_1,t_0)|\psi(t_0)\rangle = U(t,t_0)$

3) No time evolution leaves state invariant i.e. $U(t,t) = 1$

What is the form of such an operator?

Sch equation in Int. Pic.: $i|\dot\psi\rangle_I = i\,\dot U(t,t_0)|\psi(t_0)\rangle_I = H_I\,U(t,t_0)|\psi(t_0)\rangle_I$

Therefore: $\boxed{U(t,t_0) = T\exp\left[-i\int_{t_0}^{t} H_I(t')\,dt'\right] \quad \text{Dyson's Formula}}$ where $T\,O_1(t_1)O_2(t_2) = \begin{cases} O_1(t_1)O_2(t_2) & \text{if } t_1 > t_2 \\ O_2(t_2)O_1(t_1) & \text{if } t_2 > t_1 \end{cases}$

Why do we need the time ordered solution?

Excluding time ordering we have: $U(t,t_0) = \exp\left[-i\int_{t_0}^{t} H(t')\,dt'\right] = 1 - i\int_{t_0}^{t} H_I(t')\,dt' - \frac{1}{2}\left[\int_{t_0}^{t} H_I(t')\,dt'\right]^2 + \dots$

Taking time derivative:

$$\dot U(t,t_0) = -i\,H_I(t) - \frac{1}{2}\left\{\left[\int_{t_0}^{t} H_I(t')\,dt'\right]H_I(t) + H_I(t)\left[\int_{t_0}^{t} H_I(t')\,dt'\right]\right\} + \dots =$$

$$= -i\,H_I(t) - H_I(t)\left[\int_{t_0}^{t} H_I(t')\,dt'\right] - \frac{1}{2}\int_{t_0}^{t}[H_I(t'), H_I(t)]\,dt' + \dots$$

If $[H(t'), H(t)] = 0 \quad \forall t', t$ we would have: $i\dot U(t,t_0) = H_I(t)\left[1 - i\int_{t_0}^{t} H(t')\,dt' + \dots\right] = H_I(t)\,U(t,t_0)$ Satisfies Sch. Eq.

However, as $[H_I(t'), H_I(t)] \neq 0$, Sch. equation is not satisfied because of ordering issues

Claim: The time evolution operator is given by Dyson's Formula: $U(t,t_0) = T\exp\left[-i\int_{t_0}^{t} H_I(t')\,dt'\right]$

Expansion of Dyson's Formula: $U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\,H_I(t') + (-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\,H_I(t')H_I(t'') + \dots$

Proof: As t is the latest time we have:

$$i\frac{d}{dt}U(t,t_0) = i\frac{d}{dt}T\exp\left[-i\int_{t_0}^{t} dt'\,H_I(t')\right] = T\,i\frac{d}{dt}\exp\left[-i\int_{t_0}^{t} dt'\,H_I(t')\right] = T\,H_I(t)\exp\left[-i\int_{t_0}^{t} H_I(t')\,dt'\right] = H_I(t)\,U(t,t_0)$$

Examples Of Interactions

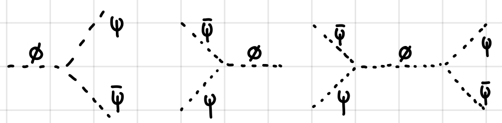
1) ϕ^4 Theory: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4$ with $\lambda \ll 1$

   By expanding ϕ^4 we will see the following terms: $(a^\dagger_{\vec p})^4$, $(a^\dagger_{\vec p})^3 a_{\vec p}$, etc.

   These create and destroy particles $\Longrightarrow$ Particle number not conserved

2) Scalar Yukawa Theory: $\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - M^2\psi^*\psi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi = \mathcal{L}_\psi + \mathcal{L}_\phi - g\psi^*\psi\phi$ with $g \ll M, m$

   While the $\psi^*\psi\phi$ interaction does not allow for the individual conservation of ϕ and ψ particles, it can be proven that the lagrangian is invariant under phase rotation of ψ leading a conserved charge Q i.e. difference between the number of ψ and anti-ψ (i.e. $\bar\psi$) particles is constant



N.B. Local minimum at $\phi = \psi = 0$ but unbounded from below for large $-g\phi$

Scattering

The interaction Hamiltonian H_{int} can be derived from $\mathcal{L}$ by computation of $\mathcal{H}_{int}$

$\mathcal{H}_{int}$ will contain several different fields, each one with a specific set of operators

As H_{int} will affect $U(t,t_0)$ (See Dyson's Formula) the different combinations of operators in the expansion will show different types of reaction

Example: Scalar Yukawa Potential

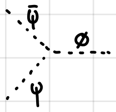
Interaction Hamiltonian: $H_{int} = g\int d^3x\, \psi^\dagger \psi \phi$

Fields:

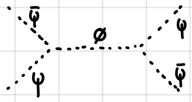
- $\phi \sim a + a^\dagger \implies$ Can create and destroy ϕ-particles i.e. mesons
- $\psi \sim b + c^\dagger \implies$ Can create $\bar{\psi}$ and destroy ψ particles i.e. fermions e.g. Nucleons
- $\psi^\dagger \sim b^\dagger + c \implies$ Can create ψ and destroy $\bar{\psi}$ particles

$Q = N_c - N_b = \text{const.}$

First Order Interaction: $c^\dagger b^\dagger a$ and $a^\dagger c b$



Second Order Interaction: $(c^\dagger b^\dagger a)(c b a^\dagger)$



Amplitudes of interactions

Initial State: $|i\rangle$ at time t_-

Final State: $|f\rangle$ at time t_+

Assumption: Assume the state $|i\rangle$ at $t_- \to -\infty$ and the state $|f\rangle$ at $t_+ \to +\infty$ to be eigenstates of the "Free Hamiltonian" H_0

The assumption is based on the idea that, prior to the interaction, the state $|i\rangle$ is formed by a set of non interacting particles that are eigenstates of H_0. They then approach each other and interact briefly. The particles the move away from each other, forming a new non-interacting state

In addition, the $|i\rangle$ and $|f\rangle$ states are expected to commute with individual number operators N, which commutes with H_0 but not H_{int}

N.B.:

- Assumption does not hold for bound states
  e.g. $e^- + p \longrightarrow H$ interaction continues in $|f\rangle$
- In QFT a particle is never truly alone due to many (virtual) excitations of vacuum

$$\boxed{\text{Amplitude:}\quad A = \lim_{t_\pm \to \pm\infty} \langle f | U(t_+, t_-) | i \rangle = \langle f | \underbrace{S}_{\text{Scattering (S) - Matrix}} | i \rangle}$$

Example: Meson Decay

$[b_{\vec p}, b^\dagger_{\vec q}] = b_{\vec p} b^\dagger_{\vec q} - b^\dagger_{\vec q} b_{\vec p} = (2\pi)^3 \delta(\vec p - \vec q)$

Consider the interaction: $b^\dagger c^\dagger a$ (1^{st} Order Int)

Initial State: $|i\rangle = \sqrt{2E_{\vec p}}\, a^\dagger_{\vec p} |0\rangle$

Final State: $|f\rangle = \sqrt{4E_{\vec q_1} E_{\vec q_2}}\, b^\dagger_{\vec q_1} c^\dagger_{\vec q_2} |0\rangle$

Considering only the 1^{st} order term: $U(t,t_0) = -i\int_{t_0}^{t} dt' H_I(t') = -i\int e^{iH_0 t'} g\, \psi^\dagger(x)\psi(x)\phi(x) e^{-iH_0 t'} d^3x\, dt' = -ig\int d^4x\, \psi^\dagger(x)\psi(x)\phi(x)$

Amplitude: $\langle f | S | i \rangle = -ig \langle f | \int d^4x\, \psi^\dagger \psi \phi | i \rangle$

Fields: $\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec k}}} \left(a_{\vec k} e^{-ik\cdot x} + a^\dagger_{\vec k} e^{ik\cdot x} \right)$ $\quad \psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec k}}} \left(b_{\vec k} e^{-ik\cdot x} + c^\dagger_{\vec k} e^{ik\cdot x} \right)$ $\quad \psi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec k}}} \left(c_{\vec k} e^{-ik\cdot x} + b^\dagger_{\vec k} e^{ik\cdot x} \right)$

It follows that:

$$\phi(x)|i\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{2E_{\vec p}}}{\sqrt{2E_{\vec k}}} \left(a_{\vec k} a^\dagger_{\vec p} e^{-ik\cdot x} + a^\dagger_k a^\dagger_{\vec p} e^{ik\cdot x} \right)|0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{2E_{\vec p}}}{\sqrt{2E_{\vec k}}} \left(a^\dagger_{\vec p} a_{\vec k} + [a_{\vec k}, a^\dagger_{\vec p}] \right) e^{-ik\cdot x}|0\rangle + \underbrace{|m_1, m_2\rangle}_{\text{2 meson state with zero overlap with } |f\rangle} = e^{-ip\cdot x}|0\rangle$$

$$\langle f|S|i\rangle = -ig\langle 0| \int d^4x \frac{d^3k_1 d^3k_2}{(2\pi)^6} \frac{\sqrt{E_{\vec q_1} E_{\vec q_2}}}{\sqrt{E_{\vec k_1} E_{\vec k_2}}} c_{\vec q_2} b_{\vec q_1} \left(c_{\vec k_1} e^{-ik_1\cdot x} + b^\dagger_{\vec k_1} e^{ik_1\cdot x} \right)\left(c^\dagger_{\vec k_2} e^{+ik_2\cdot x} + b_{\vec k_2} e^{-ik_2\cdot x} \right) e^{-ip\cdot x}|0\rangle =$$

$$= -ig\langle 0| \int d^4x \frac{d^3k_1 d^3k_2}{(2\pi)^6} \frac{\sqrt{E_{\vec q_1} E_{\vec q_2}}}{\sqrt{E_{\vec k_1} E_{\vec k_2}}} c_{\vec q_2} b_{\vec q_1} \left(c_{\vec k_1} c^\dagger_{k_2} e^{i(k_2-k_1)\cdot x} + c_{\vec k_1} b_{\vec k_2} e^{-i(k_1+k_2)\cdot x} + b^\dagger_{\vec k_1} c^\dagger_{\vec k_2} e^{i(k_1+k_2)\cdot x} + b^\dagger_{\vec k_1} b_{\vec k_2} e^{i(k_1-k_2)x} \right) e^{-ip\cdot x}|0\rangle =$$

$$\langle f|S|i\rangle = -ig\langle 0|\int d^4x \frac{d^3k_1 d^3k_2}{(2\pi)^3}\frac{\sqrt{E_{\vec{q}_1}E_{\vec{q}_2}}}{\sqrt{E_{\vec{k}_1}E_{\vec{k}_2}}}\left[c_{\vec{q}_2}b_{\vec{q}_1}[c^{\dagger}_{\vec{k}_2}, c_{\vec{k}_1}]e^{i(k_2-k_1-p)\cdot x}|0\rangle + c_{\vec{q}_2}b_{\vec{q}_1}b^{\dagger}_{\vec{k}_1}c^{\dagger}_{\vec{k}_2}e^{i(k_2+k_1-p)\cdot x}|0\rangle\right] =$$

$$= -ig\langle 0|\int d^4x \frac{d^3k_1 d^3k_2}{(2\pi)^6}\frac{\sqrt{E_{\vec{q}_1}E_{\vec{q}_2}}}{\sqrt{E_{\vec{k}_1}E_{\vec{k}_2}}}\left[(c^{\dagger}_{\vec{k}_2}c_{\vec{q}_2} + [c_{\vec{q}_2}, c_{\vec{k}_2}])(b^{\dagger}_{\vec{k}_1}b_{\vec{q}_1} + [b_{\vec{q}_1}, b^{\dagger}_{\vec{k}_2}])e^{i(k_1+k_2-p)\cdot x}|0\rangle =$$

$$= -ig\langle 0|\int d^4x\, d^3k_1 d^3k_2 \frac{\sqrt{E_{\vec{q}_1}E_{\vec{q}_2}}}{\sqrt{E_{\vec{k}_1}E_{\vec{k}_2}}}(\delta(\vec{q}_2-\vec{k}_2)\delta(\vec{q}_1-\vec{k}_1)]e^{i(k_1+k_2-p)\cdot x}|0\rangle =$$

$$= -ig\langle 0|\int d^4x\, e^{i(q_1+q_2-p)\cdot x}|0\rangle = -ig(2\pi)^4\langle 0|\delta(q_1+q_2-p)|0\rangle$$

It follows that: $\langle f|S|i\rangle = \begin{cases} -ig(2\pi)^4 & \text{if } q_1+q_2=p \\ 0 & \text{if } q_1+q_2\neq p \end{cases} \Longrightarrow$ As momentum are conserved is conserved $\langle f|S|i\rangle = -ig$

Wick's Theorem

Consider a real scalar field $\phi(x)$. It can be decomposed into:

$\phi^{+}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}a_{\vec{p}}e^{-ip\cdot x}$ "Positive Frequency Piece"

$\phi^{-}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}a^{\dagger}_{\vec{p}}e^{+ip\cdot x}$ "Negative Frequency Piece"

Note:

- Normal Ordering requires ϕ^{-} to be to the left of ϕ^{+}
-

$$\boxed{\begin{aligned} &T\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta_F(x-y) \\ &T\psi(x)\psi^{\dagger}(y) = :\psi(x)\psi^{\dagger}(y): + \Delta_F(x-y) \\ &\text{where } \Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{ik\cdot(x-y)}}{k^2-m^2+i\varepsilon} \end{aligned}}$$

Assuming $x^0 > y^0$:

$$T\phi(x)\phi(y) = \phi(x)\phi(y) = (\phi^{+}(x)+\phi^{-}(x))(\phi^{+}(y)+\phi^{-}(y)) =$$
$$= \phi^{+}(x)\phi^{+}(y) + \phi^{+}(x)\phi^{-}(y) + \phi^{-}(x)\phi^{+}(y) + \phi^{-}(x)\phi^{-}(y) =$$
$$= \phi^{+}(x)\phi^{+}(y) + \phi^{-}(y)\phi^{+}(x) + [\phi^{+}(x), \phi^{-}(y)] + \phi^{-}(x)\phi^{+}(y) + \phi^{-}(x)\phi^{-}(y) =$$
$$= \phi^{+}(x)\phi^{+}(y) + \phi^{-}(x)\phi^{+}(y) + \phi^{-}(y)\phi^{+}(x) + \phi^{-}(x)\phi^{-}(y) + [\phi^{+}(x), \phi^{-}(y)] =$$
$$= :\phi(x)\phi(y): + D(x-y)$$

Similarly, if $x^0 < y^0$:

$$T\phi(x)\phi(y) = :\phi(x)\phi(y): + D(y-x)$$

Definition: Contraction of a pair of fields in a string of operators $\dots\phi(x_1)\dots\phi(x_2)\dots$ means to replacing those operators with the Feynman Propagator
The contractions, based on previous results, are:

$$\overbrace{\phi(x)\phi(y)} = \overbrace{\psi(x)\psi^{\dagger}(y)} = \Delta_F(x-y) \quad \text{and} \quad \overbrace{\psi(x)\psi(y)} = \overbrace{\psi^{\dagger}(x)\psi^{\dagger}(y)} = 0$$

Theorem: For a collection of N fields $\phi_i = \phi(x_i)\ \forall i \in [1, N]$ we have: $T(\phi_1\dots\phi_N) = :\phi_1\dots\phi_N: + :$ All possible Contractions:

e.g. $T(\phi_1\phi_2\phi_3\phi_4) = :\phi_1\phi_2\phi_3\phi_4: + \overbrace{\phi_1\phi_2}:\phi_3\phi_4: + \overbrace{\phi_1\phi_3}:\phi_2\phi_4: + \overbrace{\phi_1\phi_4}:\phi_2\phi_3: + \overbrace{\phi_2\phi_3}:\phi_1\phi_4: + \overbrace{\phi_2\phi_4}:\phi_1\phi_3: + \overbrace{\phi_3\phi_4}:\phi_1\phi_2: +$
$+ \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_3}\,\overbrace{\phi_2\phi_4} + \overbrace{\phi_1\phi_4}\,\overbrace{\phi_2\phi_3}$

Representation of Lorentz Group

A general field $\phi^a(x)$ can transform as: $\phi^a(x) \longmapsto D[\Lambda]^a{}_b\, \phi^b(\Lambda^{-1}x)$

$D[\Lambda]$ is a representation of the Lorentz Group and thus satisfies the following properties:

- $D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2]$
- $D[\Lambda^{-1}] = D^{-1}[\Lambda]$
- $D[1] = \mathbb{1}$

The Lorentz Group is a Lie group and we thus consider the infinitesimal transformation: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ with the property $\eta^{\sigma\rho}\Lambda^\mu{}_\sigma\Lambda^\nu{}_\rho = \eta^{\mu\nu}$ s.t. $\omega^{\mu\nu} + \omega^{\nu\mu} = 0$

As $\omega^{\mu\nu}$ is antisymmetric there 6 independent components in a [4] dimensional representation

There are 6 transformations: 3 boosts + 3 rotations i.e. one for each independent element

We can define a basis of six diff. matrix to describe any transformation

Basis matrices are called generators $M^A (A = 1, \dots, 6)$ or $M^{\rho\sigma}$ (with $M^{\rho\sigma} = -M^{\sigma\rho}$)

↳ Definition of antisymmetric generators:

$$\left.\begin{aligned} (M^{\rho\sigma})^{\mu\nu} &= \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu} \\ (M^{\rho\sigma})^\mu{}_\nu &= \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu \end{aligned}\right] \quad \begin{aligned} &M^{0i}: \text{ Boost in } x^i\text{-direction} \\ &M^{ij}: \text{ Rotation in } x^i, x^j\text{-plane} \end{aligned}$$

↳ Generators obey Lie Algebra:

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau}M^{\rho\nu} - \eta^{\rho\tau}M^{\sigma\nu} + \eta^{\rho\nu}M^{\sigma\tau} - \eta^{\sigma\nu}M^{\rho\tau}$$

We can now write $\omega^\mu{}_\nu$ as a linear superposition of generators: $\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu$

The reps can then be written as: $\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$

Spinor Representation

Clifford Algebra: $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}$ where γ^μ is a matrix

which obey (4.13). For example, we may take

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

where each element is itself a 2×2 matrix, with the σ^i the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which themselves satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$. $[\sigma^i, \sigma^j] = 2\epsilon_{ijk}\sigma_k$

Properties of γ-matrices:

1) From $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$ we have:
   - $(\gamma^0)^2 = \mathbb{1}$ and $(\gamma^i)^2 = -\mathbb{1}$
   - $\gamma^\mu\gamma^\nu = -\gamma^\mu\gamma^\nu$ if $\mu \neq \nu$

2) Commutator: $S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}(\gamma^\rho\gamma^\sigma - \eta^{\rho\sigma}\mathbb{1}) \Longrightarrow S^{\mu\mu} = 0$ and $S^{\mu\nu} = \frac{1}{2}\gamma^\mu\gamma^\nu$ if $\mu \neq \nu$

$$\left.\begin{aligned} [S^{\mu\nu}, \gamma^\rho] &= \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} \\ [S^{\mu\nu}, S^{\rho\sigma}] &= \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho} \end{aligned}\right| \Longrightarrow S^{\mu\nu} \text{ Matrices satisfy Lorentz Algebra}$$

Proofs

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{4}[\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho] = \frac{1}{4}[2\gamma^\rho\gamma^\sigma - \{\gamma^\rho, \gamma^\sigma\}] = \frac{1}{2}(\gamma^\rho\gamma^\sigma - \eta^{\rho\sigma}\mathbb{1}) = \begin{cases} 0 & \text{if } \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma & \text{if } \rho \neq \sigma \end{cases}$$

$$\begin{aligned} [S^{\mu\nu}, \gamma^\rho] &= -\frac{1}{2}(\eta^{\mu\nu}\mathbb{1} - \gamma^\mu\gamma^\nu)\gamma^\rho - \frac{1}{2}\gamma^\rho[-(\eta^{\mu\nu}\mathbb{1} - \gamma^\mu\gamma^\nu)] = \\ &= -\frac{1}{2}(\gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\rho) = -\frac{1}{2}\left[\{\gamma^\rho, \gamma^\mu\}\gamma^\nu - \gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\mu\{\gamma^\nu, \gamma^\rho\} + \gamma^\mu\gamma^\rho\gamma^\nu\right] = \\ &= -\frac{1}{2}\left[2\eta^{\rho\mu}\gamma^\nu - 2\eta^{\nu\rho}\gamma^\mu\right] = \eta^{\nu\rho}\gamma^\mu - \eta^{\rho\mu}\gamma^\nu \end{aligned}$$

$$\begin{aligned} [S^{\mu\nu}, S^{\rho\sigma}] &= \frac{1}{4}(\gamma^\mu\gamma^\nu - \eta^{\mu\nu}\mathbb{1})(\gamma^\rho\gamma^\sigma - \eta^{\rho\sigma}\mathbb{1}) - \frac{1}{4}(\gamma^\rho\gamma^\sigma - \eta^{\rho\sigma}\mathbb{1})(\gamma^\mu\gamma^\nu - \eta^{\mu\nu}\mathbb{1}) = \\ &= \frac{1}{4}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\nu\eta^{\rho\sigma} - \gamma^\rho\gamma^\sigma\eta^{\mu\nu} + \eta^{\mu\nu}\eta^{\rho\sigma}\mathbb{1} - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu + \gamma^\rho\gamma^\sigma\eta^{\mu\nu} + \gamma^\mu\gamma^\nu\eta^{\rho\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma}\mathbb{1}\right] = \\ &= \frac{1}{4}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\right] = \\ &= \frac{1}{4}\left[\gamma^\mu\{\gamma^\nu, \gamma^\rho\}\gamma^\sigma - \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma - \gamma^\rho\{\gamma^\sigma, \gamma^\mu\}\gamma^\nu + \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu\right] = \\ &= \frac{1}{4}\left[\gamma^\mu\gamma^\sigma\{\gamma^\nu, \gamma^\rho\} - \{\gamma^\mu, \gamma^\rho\}\gamma^\nu\gamma^\sigma + \gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\rho\gamma^\nu\{\gamma^\sigma, \gamma^\mu\} + \gamma^\rho\gamma^\mu\{\gamma^\sigma, \gamma^\nu\} - \gamma^\rho\gamma^\mu\gamma^\nu\gamma^\sigma\right] = \\ &= \frac{1}{2}\left[\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \gamma^\nu\gamma^\sigma\eta^{\mu\rho} + \eta^{\sigma\nu}\gamma^\rho\gamma^\mu - \eta^{\sigma\mu}\gamma^\rho\gamma^\nu\right] = \\ &= \frac{1}{2}\left[(\gamma^\mu\gamma^\sigma - \eta^{\mu\sigma}\mathbb{1})\eta^{\nu\rho} - (\gamma^\nu\gamma^\sigma - \eta^{\nu\sigma})\eta^{\mu\rho} + (\gamma^\rho\gamma^\mu - \eta^{\rho\mu})\eta^{\sigma\nu} - \eta^{\sigma\mu}(\gamma^\rho\gamma^\mu - \eta^{\rho\mu}\mathbb{1})\right] = \\ &= \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\sigma\nu}S^{\rho\mu} - \eta^{\sigma\mu}S^{\rho\mu} \end{aligned}$$

Spinors

Under Lorentz Transformation we have:

Same Param to ensure same transf

$$\psi^\alpha(x) \longmapsto S[\Lambda]^\alpha{}_\beta\, \psi^\beta(\Lambda^{-1}x) \quad \text{where} \quad \Lambda = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma} M^{\rho\sigma}\right) \Longrightarrow S[\Lambda] = \exp\left(\tfrac{1}{2}\Omega_{\rho\sigma} S^{\rho\sigma}\right)$$

<u>N.B.</u> $\Omega_{\rho\sigma}$ are the same for Λ and for $S[\Lambda]$ even though $M^{\rho\sigma} \neq S^{\rho\sigma}$. This ensures that they represent the same transformation

Rotations

A rotation in x^i-x^j plane is given by $S^{ij} = \frac{1}{2}\gamma^i\gamma^j$ with $i \neq j$ $\Longrightarrow S^{ij} = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix}$

What is $\sigma^i\sigma^j$?

Pauli Matrices satisfy:

$\{\sigma^i, \sigma^j\} = \sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}$

$[\sigma^i, \sigma^j] = \sigma^i\sigma^j - \sigma^j\sigma^i = 2i\epsilon^{ijk}\sigma^k$

It follows that:

$$\left.\begin{aligned} &\sigma^j\sigma^i = 2\delta^{ij} - \sigma^i\sigma^j \\ &[\sigma^i, \sigma^j] = 2(\sigma^i\sigma^j - \delta^{ij}) = 2i\epsilon^{ijk}\sigma^k \end{aligned}\right\} \quad \sigma^i\sigma^j = i\epsilon^{ijk}\sigma^k + \delta^{ij}$$

$$\Longrightarrow S^{ij} = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} + \delta^{ij}\mathbb{1}_{4\times4} = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Then $\Omega_{\rho\sigma} = -\frac{1}{6}\epsilon_{ijk}\varphi^k$ s.t. $\Omega_{12} = -\frac{1}{6}\varphi^3$ i.e. rotation around x^3 by angle φ^3

It the follows that: $\Omega_{ij}S^{ij} = \Omega_{12}S^{12} + \Omega_{21}S^{21} + \ldots = 2(\Omega_{21}S^{21}) + \ldots = 2(\Omega_{12}S^{12} + \Omega_{13}S^{13} + \Omega_{23}S^{23}) = \frac{i}{2}\left[\varphi^3\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} + \varphi^2\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} + \varphi^1\begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}\right] = \frac{i}{2}\vec{\varphi}\cdot\vec{\sigma}$

Let's define: $\vec{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$ and $\vec{\sigma} = (S^{23}, S^{13}, S^{12})$ s.t. $S[\Lambda] = \begin{pmatrix} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix}$

Boosts

$$S^{0i} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad \text{and if } \Omega_{i0} = -\Omega_{0i} = \chi_i \text{ we have } S[\Lambda] = \begin{pmatrix} e^{+\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix}$$

<u>N.B</u> There are no finite dimensional unitary representations of the Lorentz Group. Therefore $S^\dagger[\Lambda]S[\Lambda] \neq 1$

Proof that $S^\dagger[\Lambda]S[\Lambda] \neq 1$:

$S^{-1}[\Lambda]S[\Lambda] = \mathbb{1} \Longrightarrow S^{-1}[\Lambda] = \exp(-\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma})$

Thus, $S^\dagger[\Lambda] = S^{-1}[\Lambda]$ if $(S^{\rho\sigma})^\dagger = -S^{\rho\sigma}$ i.e. $S^{\rho\sigma}$ is anti-hermitian

As $S^{\rho\sigma} \propto [\gamma^\rho, \gamma^\sigma]$, $S^{\rho\sigma}$ is anti-hermitian if all γ^μ are anti-hermitian

However:

$(\gamma^0)^2 = \mathbb{1} \Longrightarrow$ Real Eigenvalues

$(\gamma^i)^2 = -\mathbb{1} \Longrightarrow$ Imaginary Eigenvalues

If we choose γ^i to be anti-hermitian, γ^0 will be hermitian

↳ Rotations are unitary but boosts are not

There is no way to chose γ^μ such that $S^{\mu\nu}$ is anti-hermitian

In the Chiral representation: $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$

Constructing an Action

Consider the field $\psi(x)$ with adjoint $\psi^\dagger$

It follows that: $\psi(x) \longmapsto S[\Lambda]\psi(\Lambda^{-1}x)$ and $\psi^\dagger(x) \longmapsto \psi^\dagger(\Lambda^{-1}x)S^\dagger[\Lambda]$ $\Longrightarrow$ As $S^\dagger[\Lambda]S[\Lambda] \neq 1$ $\psi^\dagger(x)\psi(x)$ is not a Lorentz Scalar

As the action must be a suitable Lorentz Scalar we need to find an appropriate Lorentz Scalar

Let's consider a representation that satisfies $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$

It follows that: $\gamma^0\gamma^\mu\gamma^0 = \gamma^0\{\gamma^\mu, \gamma^0\} - (\gamma^0)^2\gamma^\mu = 2\eta^{0\mu}\gamma^0 - \gamma^\mu = (\gamma^\mu)^\dagger$

$$(S^{\mu\nu})^\dagger = \frac{1}{4}[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger] = \frac{1}{4}[\gamma^0\gamma^\nu(\gamma^0)^2\gamma^\mu\gamma^0 - \gamma^0\gamma^\mu(\gamma^0)\gamma^\nu\gamma^0] = \gamma^0[\gamma^\nu, \gamma^\mu]\gamma^0 = -\gamma^0 S^{\mu\nu}\gamma^0$$

As a result: $S^\dagger[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right) = \exp\left[-\frac{1}{2}\Omega_{\rho\sigma}\gamma^0 S^{\rho\sigma}\gamma^0\right] = \gamma^0 S^{-1}[\Lambda]\gamma^0$

Summary

$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ and $(S^{\mu\nu})^\dagger = -\gamma^0 S^{\mu\nu}\gamma^0$

It follows that:

$S^\dagger[\Lambda] = \gamma^0 S^{-1}[\Lambda]\gamma^0$

$\bar{\psi} = \psi^\dagger\gamma^0$ s.t. $\bar{\psi}\psi$ is a Lorentz Scalar

Define the Dirac Adjoint: $\bar{\psi} = \psi\gamma^0$

Claim: $\bar{\psi}(x)\psi(x)$ is a Lorentz scalar

Proof: $\bar{\psi}(x)\psi(x) = \psi^\dagger(x)\gamma^0\psi(x)$

$$\bar{\psi}(x)\psi(x) \longmapsto \psi^\dagger(\Lambda^{-1}x)S^\dagger[\Lambda]\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x) =$$
$$= \psi^\dagger(\Lambda^{-1}x)\gamma^0 S^{-1}[\Lambda](\gamma^0)^2 S[\Lambda]\psi(\Lambda^{-1}x) =$$
$$= \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}x) = \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x)$$

Claim: $\bar{\psi}\gamma^\mu\psi$ transforms like a vector

Proof:
$$\bar{\psi}\gamma^\mu\psi \longmapsto \psi^\dagger(\Lambda^{-1}x)S^\dagger[\Lambda]\gamma^\mu S[\Lambda]\psi(\Lambda^{-1}x) =$$
$$= \psi^\dagger(\Lambda^{-1}x)\gamma^0 S^{-1}[\Lambda](\gamma^0)^2\gamma^\mu S[\Lambda]\psi(\Lambda^{-1}x) =$$
$$= \psi^\dagger(\Lambda^{-1}x)\gamma^0 S^{-1}[\Lambda]\gamma^\mu S[\Lambda]\psi(\Lambda^{-1}x) =$$
$$= \bar{\psi}(\Lambda^{-1}x) S^{-1}[\Lambda]\gamma^\mu S[\Lambda]\psi(\Lambda^{-1}x) =$$
$$= \bar{\psi}(\Lambda^{-1}x)\{[S^{-1}[\Lambda], \gamma^\mu]S[\Lambda] + \gamma^\mu\}\psi(\Lambda^{-1}x)$$

If $\bar{\psi}\gamma^\mu\psi$ transforms like a vector: $\Lambda^\mu{}_\nu\gamma^\nu = \gamma^\mu + [S^{-1}[\Lambda], \gamma^\mu]S[\Lambda]$

As $\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right) \approx 1 + \frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma} + \ldots$ we have: $\Lambda^\mu{}_\nu \approx \delta^\mu{}_\nu + \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu + \ldots$

As $S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right)$ we have:

$$S[\Lambda] \approx 1 + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \ldots$$
$$S^{-1}[\Lambda] \approx 1 - \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \ldots$$

It follows that: $[S^{-1}[\Lambda], \gamma^\mu]S[\Lambda] = \left\{\overbrace{[1, \gamma^\mu]}^{=0} - \frac{1}{2}\Omega_{\rho\sigma}[S^{\rho\sigma}, \gamma^\mu] + \ldots\right\}\left[1 + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + \ldots\right] \approx -\frac{1}{2}\Omega_{\rho\sigma}[S^{\rho\sigma}, \gamma^\mu] = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu$

Then: $[S^{\rho\sigma}, \gamma] = -(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu$

As $[S^{\rho\sigma}, \gamma^\mu] = \gamma^\rho\eta^{\sigma\mu} - \gamma^\sigma\eta^{\mu\rho}$ and $(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu$ we have:

$$(M^{\rho\sigma})^\mu{}_\nu\gamma^\nu = -(\eta^{\sigma\mu}\gamma^\rho - \eta^{\rho\mu}\gamma^\sigma) = -[S^{\sigma\rho}, \gamma^\mu]$$

Thus: $\bar{\psi}(x)\gamma^\mu\psi(x) \longmapsto \Lambda^\mu{}_\nu\bar{\psi}(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x)$

Claim: $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz Tensor

The symmetric part transforms proportional to $\eta^{\mu\nu}\bar{\psi}\psi$ and the anti symm. $\sim \bar{\psi}S^{\mu\nu}\psi$

Dirac Action and Equations

Using the Lorentz scalars we can create the following action: $S = \int d^4x \; \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$

The Dirac Lagrangian is thus: $\mathcal{L}_D = \bar{\psi}(x)[i\not\partial - m]\psi(x)$ where $\not\partial = \gamma^\mu \partial_\mu$

Dirac Equations

Applying the Euler-Lagrange equation to $\mathcal{L}_D$ gives the following quantities:

$\partial\mathcal{L}/\partial\psi = -\bar{\psi}(x)m$ $\qquad$ $\partial\mathcal{L}/\partial(\partial_\alpha\psi) = \bar{\psi}(x)i\gamma^\alpha$

$\partial\mathcal{L}/\partial\bar{\psi} = (i\not\partial - m)\psi(x)$ $\qquad$ $\partial\mathcal{L}/\partial(\partial_\alpha\bar{\psi}) = 0$

As a result we have the following equations:

Dirac Equation: $(i\not\partial - m)\psi(x) = 0$

Adjoint Equation: $\bar{\psi}(x)(i\overleftarrow{\not\partial} + m) = 0$

The two equations are related by an adjoint transformation:

Adjoint: $[(i\not\partial - m)\psi(x)]^\dagger\gamma^0 = [(i\gamma^\mu\partial_\mu - m)\psi(x)]^\dagger\gamma^0 = -i(\partial_\mu\psi^\dagger(x))(\gamma^{\mu\dagger}\gamma^0) - m\psi^\dagger\gamma^0 =$

$= -i(\partial_\mu\psi^\dagger(x))(\gamma^{\mu\dagger}\gamma^0) - m\bar{\psi}(x) = -i(\partial_\mu\psi^\dagger(x))(\gamma^0\gamma^\mu\gamma^0\gamma^0) - m\bar{\psi}(x) =$

$= -\bar{\psi}(x)(i\overleftarrow{\partial}_\mu\gamma^\mu + m) = 0$

One can also show that each component of ψ satisfies the KG equation:

As $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$ so is $(i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi(x)$

Therefore: $(i\gamma^\nu\partial_\nu + m)(i\gamma^\mu\partial_\mu - m)\psi(x) = (-\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu - im\gamma^\nu\partial_\nu + im\gamma^\mu\partial_\mu - m^2)\psi(x) = -(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2)\psi(x) = 0$

It follows that: $\frac{1}{2}[(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2) + (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)]\psi(x) = (\partial^2 + m^2)\psi(x) = 0$

Symmetries of the Dirac Action/Lagrangian

The Dirac Action enjoys the following symmetries

- Spacetime translations $\quad x^\mu \longmapsto (x')^\mu = x^\mu - \epsilon^\mu$
- Lorentz transformations $\quad \psi^\alpha \longmapsto (\psi')^\alpha = S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x)$
- Internal Vector symmetry $\quad \psi \longmapsto \psi' = e^{-i\alpha}\psi$
- Axial symmetry $\quad \psi \longmapsto \psi' = e^{i\alpha\gamma^5}\psi$ and $\bar\psi \longmapsto \bar\psi' = \bar\psi e^{i\alpha\gamma^5}$

Spacetime Translations

$T: x^\mu \longmapsto x^\mu - \epsilon^\mu \implies T^{-1}: x^\mu \longmapsto x^\mu + \epsilon^\mu$

The transformations are

$\psi \longmapsto \psi'(x) = \psi(T^{-1}x)$ and $\bar\psi \longmapsto \bar\psi'(x) = \bar\psi(T^{-1}x)$ s.t. $\delta x = \epsilon^\mu$

$\psi'(x) \approx \psi(x) + (\partial_\mu\psi)\delta x = \psi(x) + \delta\psi \implies \delta\psi = (\partial_\mu\psi)\delta x = \epsilon^\mu\partial_\mu\psi$

$\bar\psi'(x) \approx \bar\psi(x) + (\partial_\mu\bar\psi)\delta x = \bar\psi(x) + \delta\bar\psi \implies \delta\bar\psi = (\partial_\mu\bar\psi)\delta x = \epsilon^\mu\partial_\mu\bar\psi$

$$\boxed{\begin{aligned}\partial_\nu\mathcal{L} &= (\partial_\nu\bar\psi)(i\gamma^\mu\partial_\mu - m)\psi(x) + \bar\psi\partial_\nu\left[(i\gamma^\mu\partial_\mu - m)\psi\right] = \\ &= (\partial_\nu\bar\psi)(i\gamma^\mu\partial_\mu - m)\psi(x) + \bar\psi(i\gamma^\mu\partial_\mu - m)(\partial_\nu\psi) =\end{aligned}}$$

The Lagrangian is: $\mathcal{L} \longmapsto \mathcal{L}'$

$$\begin{aligned}\mathcal{L}' &= \bar\psi'(x)(i\gamma^\mu\partial_\mu - m)\psi'(x) = [\bar\psi(x) + \delta\bar\psi](i\gamma^\mu\partial_\mu - m)[\psi(x) + \delta\psi] = \\ &= \mathcal{L} + \bar\psi(x)(i\gamma^\mu\partial_\mu - m)\delta\psi + \delta\bar\psi(i\gamma^\mu\partial_\mu - m)\psi(x) + \delta\bar\psi(i\gamma^\mu\partial_\mu - m)\delta\psi = \\ &= \mathcal{L} + \bar\psi(x)(i\gamma^\mu\partial_\mu - m)\epsilon^\nu\partial_\nu\psi + \epsilon^\nu\partial_\nu\bar\psi(i\gamma^\mu\partial_\mu - m)\psi(x) + \epsilon^\alpha\partial_\alpha\bar\psi(i\gamma^\mu\partial_\mu - m)\epsilon^\beta\partial_\beta\psi = \\ &= \mathcal{L} + \epsilon^\nu\partial_\nu\mathcal{L} + \epsilon^\beta\partial_\beta\bar\psi(i\gamma^\mu\partial_\mu - m)\epsilon^\gamma\partial_\gamma\psi = \mathcal{L} + \epsilon^\nu\partial_\nu\mathcal{L} = \mathcal{L} + \delta\mathcal{L}\end{aligned}$$

It follows that: $\delta\mathcal{L} = \partial_\mu F^\mu = \epsilon^\nu\partial_\nu\mathcal{L} \implies F^\mu = \delta^\mu_\nu\epsilon^\nu\mathcal{L}$

Just a scalar component so we can move it around

The conserved current is: $j^\mu = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)}\delta\bar\psi - F^\mu = \bar\psi i\gamma^\mu\epsilon^\nu\partial_\nu\psi - \delta^\mu{}_\nu\epsilon^\nu\mathcal{L}$ s.t. $\partial_\mu j^\mu = 0$

As ϵ^ν is const. we can write $j^\mu = \epsilon^\nu\left[\bar\psi i\gamma^\mu\partial_\nu\psi - \delta^\mu{}_\nu\mathcal{L}\right]$ and as $\partial_\mu j^\mu = 0$, we can remove the const. ϵ^ν entirely

We can thus write the following tensor:

Energy Momentum Tensor: $T^\mu{}_\nu = \bar\psi i\gamma^\mu\partial_\nu\psi - \delta^\mu{}_\nu\mathcal{L} \overset{\text{Using Dirac Equation}}{=} \bar\psi i\gamma^\mu\partial_\nu\psi$

Lorentz Transformations

$x^\mu \longmapsto (x')^\mu = \Lambda^\mu{}_\nu x^\nu$ $\quad$ where $\Lambda = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}M^{\rho\sigma}\right]$ s.t. $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \overbrace{\tfrac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu}^{\omega^\mu{}_\nu} + \dots$

$\psi^\alpha \longmapsto (\psi')^\alpha = S[\Lambda]^\alpha{}_\beta\psi^\beta(\Lambda^{-1}x)$ $\quad$ where $S[\Lambda] = \exp\left[\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right]$ s.t. $S[\Lambda]^\alpha{}_\beta = \delta^\alpha{}_\beta + \frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta + \dots$

$\bar\psi^\alpha \longmapsto (\bar\psi')^\alpha = \bar\psi^\beta(\Lambda^{-1}x)S[\Lambda]^\alpha{}_\beta$

Note: $(M^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu \implies \omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(\eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu) = \frac{1}{2}(\Omega^\mu{}_\nu - \Omega_\nu{}^\mu) \implies \omega^{\mu\nu} = \Omega^{\mu\nu}$

It follows that, for small transformations, the fields transform as:

$$\begin{aligned}(\psi')^\alpha &\approx \left[\delta^\alpha{}_\beta + \tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\right]\psi^\beta\left((\delta^\mu{}_\nu - \omega^\mu{}_\nu)x^\nu\right) \approx \left[\delta^\alpha{}_\beta + \tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\right]\left[\psi^\beta(x) + \delta x(\partial_\mu\psi^\beta)\right] = \\ &= \left[\delta^\alpha{}_\beta + \tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\right]\left[\psi^\beta(x) - \omega^\mu{}_\nu x^\nu(\partial_\mu\psi^\beta)\right] = \\ &= \psi^\alpha(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\psi^\alpha + \tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\psi^\beta(x) + \dots = \\ &\approx \psi^\alpha(x) - \omega^\mu{}_\nu x^\nu\partial_\mu\psi^\alpha + \tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\psi^\beta = \psi^\alpha(x) + \delta\psi^\alpha\end{aligned}$$

$$\begin{aligned}\delta\psi^\alpha &= -\omega^\mu{}_\nu x^\nu\partial_\mu\psi^\alpha + \tfrac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta\psi^\beta = -\tfrac{1}{2}\Omega_{\rho\sigma}\left[(M^{\rho\sigma})^\mu{}_\nu x^\nu\partial_\mu\psi^\alpha - (S^{\rho\sigma})^\alpha{}_\beta\psi^\beta\right] = \\ &= -\omega^{\mu\nu}\left[x_\nu\partial_\mu\psi^\alpha - \tfrac{1}{2}(S_{\mu\nu})^\alpha{}_\beta\psi^\beta\right]\end{aligned}$$

As Lagrangian is Lorentz Invariant: $\delta\mathcal{L} = 0 = \partial_\mu F^\mu$

As $\partial\mathcal{L}/\partial(\partial_\mu\bar\psi) = 0$ we don't care about $\delta\bar\psi$

The conserved current is thus:

$$\begin{aligned}j^\mu &= \bar\psi i\gamma^\mu\delta\psi = -\omega^{\rho\sigma}\bar\psi i\gamma^\mu\left[x_\sigma\partial_\rho\psi - \tfrac{1}{2}(S_{\rho\sigma})\psi\right] = \\ &= -\omega^{\rho\sigma}\left[x_\sigma\bar\psi i\gamma^\mu\partial_\rho\psi - \tfrac{1}{2}\bar\psi i\gamma^\mu(S_{\rho\sigma})\psi\right] = \\ &= -\omega^{\rho\sigma}\left[x_\sigma T^\mu{}_\rho - \tfrac{1}{2}\bar\psi i\gamma^\mu(S_{\rho\sigma})\psi\right]\end{aligned}$$

As $\omega^{\mu\nu} = -\omega^{\nu\mu}$, $S^{\sigma\rho} = -S_{\sigma\rho}$ we have that: $j^\mu = \omega^{\sigma\rho}\left[x_\rho T^\mu{}_\sigma - \frac{1}{2}\bar\psi i\gamma^\mu S_{\sigma\rho}\psi\right] = -\omega^{\rho\sigma}\left[x_\rho T^\mu{}_\sigma + \frac{1}{2}\bar\psi i\gamma^\mu S_{\rho\sigma}\psi\right]$

As both currents are conserved we can sum them to get a new conserved current $\mathcal{J}^\mu$

Thus:

$$\begin{aligned}(\mathcal{J}^\mu)^{\rho\sigma} &= \omega^{\sigma\rho}\left[x_\rho T^\mu{}_\sigma - \tfrac{1}{2}\bar\psi i\gamma^\mu S_{\sigma\rho}\psi\right] + \omega^{\rho\sigma}\left[x_\sigma T^\mu{}_\rho - \tfrac{1}{2}\bar\psi i\gamma^\mu (S_{\rho\sigma})\psi\right] = \\ &= -\omega^{\rho\sigma}\left[x_\rho T^\mu{}_\sigma + \tfrac{1}{2}\bar\psi i\gamma^\mu S_{\rho\sigma}\psi\right] + \omega^{\rho\sigma}\left[x_\sigma T^\mu{}_\rho - \tfrac{1}{2}\bar\psi i\gamma^\mu S_{\rho\sigma}\psi\right] = \\ &= -\omega^{\rho\sigma}\left[x_\rho T^\mu{}_\sigma - x_\sigma T^\mu{}_\rho + \bar\psi i\gamma^\mu S_{\rho\sigma}\psi\right]\end{aligned}$$

Note: Compared to the K-G Lagrangian, the additional term of $\bar\psi i\gamma^\mu S^{\rho\sigma}\psi$ provides the single particle states with internal angular momentum $s = 1/2$

Tong has a (-) here

We can write: $(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} + \bar\psi i\gamma^\mu S^{\rho\sigma}\psi$

Internal Vector Symmetry

Vector symmetry: Left and right handed fermions are rotated in same direction

Phase rotation of the spinor: $\psi \longmapsto \psi' = e^{-i\alpha}\psi = \left(1 - i\alpha + \frac{1}{2}(-i\alpha)^2 + \dots\right)\psi$

For small transformations: $\psi' \approx \psi - i\alpha\psi$ and $\bar\psi' \approx \bar\psi + \bar\psi i\alpha$ $\Longrightarrow$ $\delta\psi = -i\alpha\psi$ and $\delta\bar\psi = \bar\psi i\alpha$

It follows that: $j^\mu_V = -\bar\psi\gamma^\mu\alpha\psi$ or $j^\mu_V = \bar\psi\gamma^\mu\psi$

$\partial_\mu j^\mu = (\partial_\mu\bar\psi)\gamma^\mu\psi + \bar\psi\gamma^\mu(\partial_\mu\psi) = im\bar\psi\psi - im\bar\psi\psi = 0$ Thanks to E.O.M.

Conserved Charge: $Q = \int d^3x\, j^0 = \int d^3x\, \bar\psi\gamma^0\psi = \int d^3x\, \psi^\dagger\psi$ $\Longrightarrow$ Electric Charge / Particle Number of fermions

Axial Symmetry

Axial symmetry: Left and Right-handed fermions are rotated in same direction i.e. $\psi \longrightarrow e^{i\alpha\gamma^5}\psi$ and $\bar\psi \longrightarrow \bar\psi e^{i\alpha\gamma^5}$

It follows that: $\delta\psi = i\alpha\gamma^5\psi$ and $\delta\bar\psi = \bar\psi i\alpha\gamma^5$

For small rotations: $\mathcal{L}' \approx \bar\psi(1 + i\alpha\gamma^5)(i\gamma^\mu\partial_\mu - m)(1 + i\alpha\gamma^5)\psi =$

$$\begin{aligned}&= \mathcal{L} + \bar\psi(i\gamma^\mu\partial_\mu - m)i\alpha\gamma^5\psi + \bar\psi i\alpha\gamma^5(i\gamma^\mu\partial_\mu - m)\psi + \overset{\approx 0}{\alpha^2}\left[\bar\psi i\gamma^5(i\gamma^\mu\partial_\mu - m)i\psi\right] = \\ &\approx \mathcal{L} - i\alpha\left[\bar\psi\gamma^\mu\gamma^5\partial_\mu\psi + \bar\psi\gamma^5\gamma^\mu\partial_\mu\psi - 2m\bar\psi\gamma^5\psi\right] = \\ &= \mathcal{L} - i\alpha\left[\bar\psi\{\gamma^\mu,\gamma^5\}\partial_\mu\psi - 2m\bar\psi\gamma^5\psi\right] = \\ &= \mathcal{L} + 2im\alpha\bar\psi\gamma^5\psi = \mathcal{L} + \delta\mathcal{L}\end{aligned}$$

$\delta\mathcal{L} = 0$ if $m = 0$ $\Longrightarrow$ It is a symmetry of the Lagrangian for massless particles

The conserved current is: $j^\mu_A = \bar\psi\gamma^\mu\gamma^5\psi$

However, this symmetry does not survive the quantization process

Plane Wave solutions to Dirac Equation

The Dirac Equation is: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

As it is a 1st-order differential equation we expect a solution of the form: $\psi = u(\vec{p})e^{-ip\cdot x}$

To find the complete solution we must find $u(\vec{p})$ which must:

- be a 4-component spinor as γ^μ is a 4×4 matrix
- depend on 3-momentum $\vec{p}$ as the energy (i.e. p^0) depends on m and $\vec{p}$

By substitution: $(i\not{\partial} - m)\psi = (i\gamma^\mu\partial_\mu - m)u(\vec{p})e^{-ip_\alpha x^\alpha} = (\gamma^\mu p_\mu - m)u(\vec{p})e^{-ip_\alpha x^\alpha} = 0 \Longrightarrow (\gamma^\mu p_\mu - m)u(\vec{p}) = 0$

The Dirac representation is: $\gamma^0 = \text{antidiag}(\mathbb{1}_{2\times2}, \mathbb{1}_{2\times2})$ and $\gamma^i = \text{antidiag}(\sigma^i, -\sigma^i)$

Therefore: $(\gamma^0 p_0 + \gamma^i p_i - m\mathbb{1}_{4\times4})u(\vec{p}) = 0 \Longrightarrow [\text{antidiag}(p_\mu\sigma^\mu, p_\mu\bar{\sigma}^\mu) - \text{diag}(m, m)]\,u(\vec{p}) = 0$ where $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i) = \gamma^0(\sigma^+)^\mu\gamma^0$

If we write $u(\vec{p}) = (u_1, u_2)$ where u_1 and u_2 are 2 component spinors we can write:

Dirac Equation: $\begin{bmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & -m \end{bmatrix}\begin{bmatrix} u_1(\vec{p}) \\ u_2(\vec{p}) \end{bmatrix} = 0 \Longrightarrow \begin{cases} p_\mu\sigma^\mu u_2 - m u_1 = 0 & (1) \\ p_\mu\bar{\sigma}^\mu u_1 - m u_2 = 0 & (2) \end{cases}$

we now evaluate the following: $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p_\mu\sigma^\mu)(p_\nu\bar{\sigma}^\nu) = p_\mu p_\nu \sigma^\mu\bar{\sigma}^\nu = (p^0)^2 - p_i p_j\sigma^i\sigma^j = (p^0)^2 - (p^i)^2 = E^2 - \vec{p}^2 = m^2$

It follows that: $(p\cdot\bar{\sigma})(p_\mu\sigma^\mu u_2 - m u_1) = -m(p_\mu\bar{\sigma}^\mu u_1 - m u_2) = 0$ s.t. (1) implies (2) and viceversa.

It follows that $m u_1 = (p\cdot\sigma)u_2$ and $(p\cdot\bar{\sigma})u_1 = m u_2 \xRightarrow{(p\cdot\sigma)(p\cdot\bar{\sigma}) = m^2}$ Ansatz: $u_1(\vec{p}) = A(p\cdot\sigma)\varepsilon'$ and $u_2 = A m\varepsilon'$ where A is a constant

Therefore, any spinor in the form $u(\vec{p}) = A[(p\cdot\sigma)\varepsilon', m\varepsilon']^T$

To impose symmetry we set $A = m^{-1}$ and $\varepsilon' = \sqrt{p\cdot\bar{\sigma}}\,\varepsilon$

It follows that:

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\varepsilon \\ \sqrt{p\cdot\bar{\sigma}}\,\varepsilon \end{pmatrix}$$ where ε is 2-component constant spinor s.t. $\varepsilon^\dagger\varepsilon = 1$ and the state is normalised

Similarly, we can find solutions using the Ansatz $\psi = v(\vec{p})e^{ip\cdot x}$ which must satisfy $(\gamma^\mu p_\mu + m)v(\vec{p})$

Therefore:

$$v(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix}$$ where η is 2-component constant spinor s.t. $\eta^\dagger\eta = 1$ and the state is normalised

To determine solutions one could also write the explicit 4×4 Matrix

Positive and negative frequency solutions

The terms of the kind $u(\vec{p})e^{-ipx}$ which oscillate in time according to $\sim e^{-iEt}$ are the <u>positive frequency solutions</u>

The terms of the kind $v(\vec{p})e^{ipx}$ which oscillate in time according to $\sim e^{+iEt}$ are the <u>negative frequency solutions</u>

Examples of Plane Wave Solutions

Consider the case in which $\vec{p}=0$ s.t. $p=(m,\vec{0}) \implies (p\cdot\sigma)=(p\cdot\bar{\sigma})=m$

The spinor solutions are:

$$u_+(\vec{p})=\sqrt{m}\begin{pmatrix}\varepsilon\\ \varepsilon\end{pmatrix} \quad \text{and} \quad u_-(\vec{p})=\sqrt{m}\begin{pmatrix}\eta\\ -\eta\end{pmatrix}$$

Lorentz transformation acting on ψ: $\psi(x)\longmapsto S[\Lambda]\psi(\Lambda^{-1}x)=\left[S[\Lambda]u_\pm(\vec{p})\right]e^{\mp p(\Lambda^{-1}x)}$

Therefore, a spinor $u(\vec{p})$ transforms as: $u_\pm(\vec{p})\longmapsto S[\Lambda]u_\pm(\vec{p})$

① Rotations:

A rotation is represented by $S[\Lambda]=\begin{pmatrix}e^{+i\vec{\theta}\cdot\vec{\sigma}/2} & 0\\ 0 & e^{+i\vec{\theta}\cdot\vec{\sigma}/2}\end{pmatrix}$

Therefore, the spinor fields are transformed as: $\varepsilon\longmapsto e^{i\vec{\theta}\cdot\vec{\sigma}/2}\varepsilon$ and $\pm\eta\longmapsto\pm e^{i\vec{\theta}\cdot\vec{\sigma}/2}\eta$

In terms of particles, we can see that the spinor fields describe the spin of the particle. In fact, in the Quantum Mechanical interpretation, a particle (in this case a field, we are yet to quantize) has spin up/down in a specific direction if the state (spinor field) is an eigenvector of the corresponding Pauli Matrix and has eigenvalue ± 1 respectively e.g. $\varepsilon^T=(1,0)$ has spin up along $\hat{z}$ while $\varepsilon^T=(0,1)$ has spin down along $\hat{z}$

② Boosting

Consider the spin-up state above

Now, boost it along x^3 to a frame in which it has $p=(E,0,0,p)$

It follows that: $(p\cdot\sigma)=(E-p^3\sigma^3)$ and $(p\cdot\bar{\sigma})=(E^3+p^3\sigma^3)$

As $\sqrt{\varepsilon^T}=(1,0)$ it follows that:

$$u(\vec{p})=\begin{pmatrix}\sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix}\\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \xmapsto{m\to 0} u(\vec{p})=\sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Similarly, if $\varepsilon^T=(0,1)$ we have: (Note: eigenvalue is -1)

$$u(\vec{p})=\begin{pmatrix}\sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix}\\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \xmapsto{m\to 0} u(\vec{p})=\sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Helicity

Helicity: Projection of ang. momentum along direction of momentum

↳ Operator: $h=(\vec{P}\cdot\vec{S})(\hat{P}/P) \implies h=\frac{i}{2}\epsilon_{ijk}\hat{p}^i S^{jk}=\frac{1}{2}\hat{p}_i\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}$

By applying to $u(\vec{p})=\sqrt{2E}\,(0\,0\,1\,0)^T$ we get $h=+1/2 \implies \varepsilon^T=(1,0)$ is Right Handed

By applying to $u(\vec{p})=\sqrt{2E}\,(0\,1\,0\,0)^T$ we get $h=-1/2 \implies \varepsilon^T=(0,1)$ is Left Handed

Inner And Outer Products

Define: $\varepsilon^1 = (1,0)^T$ and $\varepsilon^2 = (0,1)^T$

↳ ε^s, η^s $(s=1,2)$ form a basis for the spinors $\Longrightarrow (\varepsilon^r)^\dagger \varepsilon^s = (\eta^r)^\dagger \eta^s = \delta^{rs}$

Therefore: $u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\varepsilon^s \\ \sqrt{p\cdot\bar\sigma}\,\varepsilon^s \end{pmatrix}$ $\qquad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix}$

Inner Products

In previous sections we saw that only $\bar u(\vec p)\cdot u(\vec p)$ can be Lorentz Invariant

However, $u^\dagger(\vec p)\cdot u(\vec p)$ will be important for quantisation

$$u^{r\dagger}(\vec p)\cdot u^s(\vec p) = (\varepsilon^{r\dagger}\sqrt{p\cdot\sigma},\ \varepsilon^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} \sqrt{p\cdot\sigma}\,\varepsilon^s \\ \sqrt{p\cdot\bar\sigma}\,\varepsilon^s \end{pmatrix} = \varepsilon^{r\dagger}(p\cdot\sigma)\varepsilon^s + \varepsilon^{r\dagger}(p\cdot\bar\sigma)\varepsilon^s = \varepsilon^{r\dagger}[p\cdot(\sigma+\bar\sigma)]\varepsilon^s = 2p_0\,\varepsilon^{r\dagger}\varepsilon^s = 2p_0\,\delta^{rs}$$

$$\bar u^r(\vec p)\cdot u^s(\vec p) = u^{r\dagger}(\vec p)\gamma^0 u^s(\vec p) = (\varepsilon^{r\dagger}\sqrt{p\cdot\sigma},\ \varepsilon^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\varepsilon^s \\ \sqrt{p\cdot\bar\sigma}\,\varepsilon^s \end{pmatrix} = 2\varepsilon^{r\dagger}\sqrt{(p\cdot\sigma)(p\cdot\bar\sigma)}\,\varepsilon^s = 2m\,\delta^{rs}$$

$$\bar u^r(\vec p)\cdot v^s(\vec p) = u^{r\dagger}(\vec p)\gamma^0 v^s(\vec p) = (\varepsilon^{r\dagger}\sqrt{p\cdot\sigma},\ \varepsilon^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix} = \varepsilon^{r\dagger}\left[\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} - \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}\right]\eta^s = 0$$

$$u^{r\dagger}(\vec p)\cdot v^s(\vec q) = (\varepsilon^{r\dagger}\sqrt{p\cdot\sigma},\ \varepsilon^{r\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} \sqrt{q\cdot\sigma}\,\eta^s \\ -\sqrt{q\cdot\bar\sigma}\,\eta^s \end{pmatrix} = \varepsilon^{r\dagger}\left[\sqrt{(p\cdot\sigma)(q\cdot\sigma)} - \sqrt{(p\cdot\bar\sigma)(q\cdot\bar\sigma)}\right]\eta^s \Longrightarrow u^{r\dagger}(\vec p)\cdot v^s(\vec q) = 0 \text{ if } \vec q = -\vec p$$

Similarly:

$v^{r\dagger}(\vec p)\cdot v^s(\vec p) = 2p_0\delta^{rs}$

$\bar v^r(\vec p)\cdot v^s(\vec p) = -2m\delta^{rs}$

$\bar v^r(\vec p)\cdot u^s(\vec p) = 0$

$v^{r\dagger}(\vec p)\cdot v^s(-\vec p) = 0$

Outer Products

$$\sum_{s=1}^{2} u^s(\vec p)\,\bar u^s(\vec p) = \sum_{s=1}^{2}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\varepsilon^s \\ \sqrt{p\cdot\bar\sigma}\,\varepsilon^s \end{pmatrix}(\varepsilon^{s\dagger}\sqrt{p\cdot\sigma},\ \varepsilon^{s\dagger}\sqrt{p\cdot\bar\sigma})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sum_{s=1}^{2}\begin{pmatrix} (p\cdot\sigma)\varepsilon^s\varepsilon^{\dagger s} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}\,\varepsilon\varepsilon^{s\dagger} \\ \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}\,\varepsilon^s\varepsilon^{\dagger s} & (p\cdot\bar\sigma)\varepsilon\varepsilon^{s\dagger} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix}\sum_{s=1}^{2}\varepsilon\varepsilon^{s\dagger} = \begin{pmatrix} m & \overset{p_\mu\sigma^\mu}{\overbrace{p\cdot\sigma}} \\ \underset{-p_\mu\sigma^\mu}{\underbrace{p\cdot\bar\sigma}} & m \end{pmatrix} = \not{p} + m$$

Similarly: $\sum_{s=1}^{2} v^s(\vec p)\,\bar v^s(\vec p) = \not{p} - m$

Summary

$u^{r\dagger}(\vec p)\cdot u^s(\vec p) = 2p^0\delta^{rs}$ $\qquad v^{r\dagger}(\vec p)\cdot v^s(\vec p) = 2p_0\delta^{rs}$

$\bar u^r(\vec p)\cdot u^s(\vec p) = 2m\delta^{rs}$ $\qquad \bar v^r(\vec p)\cdot v^s(\vec p) = -2m\delta^{rs}$

$\bar u^r(\vec p)\cdot v^s(\vec p) = 0$ $\qquad \bar v^r(\vec p)\cdot u^s(\vec p) = 0$

$u^{r\dagger}(\vec p)\cdot v^s(-\vec p) = 0$ $\qquad v^{r\dagger}(\vec p)\cdot v^s(-\vec p) = 0$

$\boxed{\sum_{s=1}^{2} u^s(\vec p)\,\bar u^s(\vec p) = \not{p} + m \qquad \sum_{s=1}^{2} v^s(\vec p)\,\bar v^s(\vec p) = \not{p} - m}$ $\Longrightarrow$ Very important for things that do not depend on spin s.t. we need to consider all spin contributions

Quantizing Dirac Field

The Dirac Lagrangian density is: $\mathcal{L} = \bar{\psi}(x)(i\not{\partial} - m)\psi(x) = i\bar{\psi}\gamma^0\dot{\psi} + i\bar{\psi}\gamma^j\partial_j\psi - \bar{\psi}m\psi$

The field satisfies the Dirac Equation: $(i\not{\partial} - m)\psi(x) = i\gamma^0\partial_0\psi + i\gamma^i\partial_i\psi - m\psi = 0$

We can thus compute the hamiltonian as:

$\pi(x) = (\partial\mathcal{L}/\partial\dot{\psi}) = i\bar{\psi}\gamma^0 = i\psi^\dagger$

$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi = i\bar{\psi}\gamma^0\partial_0\psi = i\psi^\dagger\partial_0\psi$

As we have already seen, the Dirac equation allows for 4 different plane wave solutions: $u^s(\vec{p})e^{-ip\cdot x}$, $v^s(\vec{p})e^{ip\cdot x}$, $u^r(\vec{p})e^{-ip\cdot x}$ and $v^r(\vec{p})e^{ip\cdot x}$

The 4 solutions represent the positive and negative frequency solutions for spin up and down respectively

It follows that the fields can be written as operators in the following way:

Heisenberg picture i.e. $i\,\partial\psi/\partial t = [\psi, H]$

$$\psi(x) = \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s(\vec{p})e^{-ip\cdot x} + c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{+ip\cdot x}\right]$$

$$\psi^\dagger(x) = \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^{s\dagger}_{\vec{p}}u^{s\dagger}(\vec{p})e^{+ip\cdot x} + c^s_{\vec{p}}v^{s\dagger}(\vec{p})e^{-ip\cdot x}\right]$$

Schrödinger picture:

$$\psi(\vec{x}) = \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s(\vec{p})e^{+i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right]$$

$$\psi^\dagger(\vec{x}) = \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^{s\dagger}_{\vec{p}}u^{s\dagger}(\vec{p})e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}}v^{s\dagger}(\vec{p})e^{i\vec{p}\cdot\vec{x}}\right]$$

The $\pm$ sign difference in the exponents between the two pictures is due to the metric

The summation over s ensures that ψ creates both spin up and down particles corresponding to the $v^s(\vec{p})$ spinor while it annihilates both spin up and down particles associated with spinor $u^s(\vec{p})$. Viceversa for $\psi^\dagger$

Explicit computation of the Hamiltonian:

In Heisenberg's picture we can use: $\mathcal{H} = i\psi^\dagger\partial_0\psi$

We know that:

$$\partial_0\psi = -\sum_s\int\frac{d^3p}{(2\pi)^3}\frac{iE_{\vec{p}}}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s(\vec{p})e^{-ip\cdot x} - c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{ip\cdot x}\right]$$

$$\mathcal{H} = \sum_s\sum_r\int\frac{d^3p\,d^3q}{(2\pi)^3}\frac{E_{\vec{p}}}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left[b^{r\dagger}_{\vec{q}}b^s_{\vec{p}}u^{r\dagger}(\vec{q})u^s(\vec{p})e^{-i(p-q)\cdot x} - b^{r\dagger}_{\vec{q}}c^{s\dagger}_{\vec{p}}u^{r\dagger}(\vec{q})v^s(\vec{p})e^{i(p+q)\cdot x} + c^r_{\vec{q}}b^s_{\vec{p}}v^{s\dagger}(\vec{q})u^s(\vec{p})e^{-i(p+q)\cdot x} - c^r_{\vec{q}}c^{s\dagger}_{\vec{p}}v^{r\dagger}(\vec{q})v^s(\vec{p})e^{i(p-q)\cdot x}\right] =$$

$$H = \int d^3x\,\mathcal{H} = \sum_s\sum_r\int d^3x\frac{d^3p\,d^3q}{(2\pi)^3}\frac{E_{\vec{p}}}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left[b^{r\dagger}_{\vec{q}}b^s_{\vec{p}}u^{r\dagger}(\vec{q})u^s(\vec{p})e^{-i(p-q)\cdot x} - c^r_{\vec{q}}c^{s\dagger}_{\vec{p}}v^{r\dagger}(\vec{q})v^s(\vec{p})e^{i(p-q)\cdot x} - b^{r\dagger}_{\vec{q}}c^{s\dagger}_{\vec{p}}u^{r\dagger}(\vec{q})v^s(\vec{p})e^{i(p+q)\cdot x} + c^r_{\vec{q}}b^s_{\vec{p}}v^{r\dagger}(\vec{q})u^s(\vec{p})e^{-i(p+q)\cdot x}\right] =$$

$$= \sum_s\sum_r\int\frac{d^3p\,d^3q}{(2\pi)^3}\frac{E_{\vec{p}}}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left[\left(b^{r\dagger}_{\vec{q}}b^s_{\vec{p}}u^{r\dagger}(\vec{q})u^s(\vec{p})e^{-i(p_0-q_0)t} - c^r_{\vec{q}}c^{s\dagger}_{\vec{p}}v^{r\dagger}(\vec{q})v^s(\vec{p})e^{i(p_0-q_0)t}\right)\delta(\vec{p}-\vec{q}) + \left(c^r_{\vec{q}}b^s_{\vec{p}}v^{r\dagger}(\vec{q})u^s(\vec{p})e^{-i(p_0+q_0)t} - b^{r\dagger}_{\vec{q}}c^{s\dagger}_{\vec{p}}u^{r\dagger}(\vec{q})v^s(\vec{p})e^{i(p_0+q_0)t}\right)\delta(\vec{p}+\vec{q})\right] =$$

$$= \sum_s\sum_r\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[b^{r\dagger}_{\vec{p}}b^s_{\vec{p}}u^{r\dagger}(\vec{p})u^s(\vec{p}) - c^r_{\vec{p}}c^{s\dagger}_{\vec{p}}v^{r\dagger}(\vec{p})v^s(\vec{p}) + c^r_{-\vec{p}}b^s_{\vec{p}}v^{s\dagger}(-\vec{p})u^s(\vec{p})e^{-i2p_0t} - b^{r\dagger}_{-\vec{p}}c^{s\dagger}_{\vec{p}}u^{r\dagger}(-\vec{p})v^s(\vec{p})e^{+i2p_0t}\right] =$$

$$= \sum_s\sum_r\int\frac{d^3p}{(2\pi)^3}p_0\delta^{rs}\left[b^{r\dagger}_{\vec{p}}b^s_{\vec{p}} - c^r_{\vec{p}}c^{s\dagger}_{\vec{p}}\right] = \sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left[b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} - c^s_{\vec{p}}c^{s\dagger}_{\vec{p}}\right]$$

Therefore the Hamiltonian is:

Not Normal-Ordered: $H = \sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left[b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} - c^s_{\vec{p}}c^{s\dagger}_{\vec{p}}\right]$

Normal Ordered:

- Commutation: $H = \sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left[b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} - c^{s\dagger}_{\vec{p}}c^s_{\vec{p}} - [c^s_{\vec{p}}, c^{s\dagger}_{\vec{p}}]\right]$
- Anticommutation: $H = \sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left[b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} + c^{s\dagger}_{\vec{p}}c^s_{\vec{p}} - \{c^s_{\vec{p}}, c^{s\dagger}_{\vec{p}}\}\right]$

Canonical Quantization

If we define the fields to be operators obeying canonical commutation relations we have: $[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = [\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = 0$ and $[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y})$

Claim:

$$\begin{aligned}&[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = [\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = 0\\ &[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y})\end{aligned} \quad \Longleftarrow \quad \begin{aligned}&[b^r_{\vec{p}}, b^s_{\vec{q}}] = [b^{r\dagger}_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = [c^r_{\vec{p}}, c^s_{\vec{q}}] = \dots = 0\\ &[b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = -[c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}] = (2\pi)^3\delta^{rs}\,\delta(\vec{p}-\vec{q})\end{aligned}$$

Attention to minus sign!

Proof:

$$\psi_\alpha(\vec{x})\psi^\dagger_\beta(\vec{y}) = \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\sum_{s,r}\Big[b^s_{\vec{p},\alpha} b^{r\dagger}_{\vec{q},\beta} u^s_\alpha(\vec{p}) u^{r\dagger}_\beta(\vec{q}) e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + b^s_{\vec{p},\alpha} c^r_{\vec{q},\beta} u^s_\alpha(\vec{p}) v^{r\dagger}_\beta(\vec{q}) e^{i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + c^{s\dagger}_{\vec{p},\alpha} b^{r\dagger}_{\vec{q},\beta} v^s_\alpha(\vec{p}) u^{r\dagger}_\beta(\vec{q}) e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + c^{s\dagger}_{\vec{p}\alpha} c^r_{\vec{q},\beta} v^s_\alpha(\vec{p}) v^{r\dagger}_\beta(\vec{q}) e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\Big]$$

$$\psi^\dagger_\beta(\vec{y})\psi_\alpha(\vec{x}) = \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\sum_{s,r}\Big[b^{r\dagger}_{\vec{q},\beta} b^s_{\vec{p},\alpha} u^{r\dagger}_\beta(\vec{q}) u^s_\alpha(\vec{p}) e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + b^{r\dagger}_{\vec{p},\beta} c^{s\dagger}_{\vec{p},\alpha} u^{r\dagger}_\beta(\vec{q}) u^s_\alpha(\vec{p}) e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + c^r_{\vec{q},\beta} b^s_{\vec{p},\alpha} v^{r\dagger}_\beta(\vec{q}) u^s_\alpha(\vec{p}) e^{i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + c^r_{\vec{q},\beta} c^{s\dagger}_{\vec{p},\alpha} v^{r\dagger}_\beta(\vec{q}) v^s_\alpha(\vec{p}) e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\Big]$$

It follows that:

$$[\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\sum_{s,r}\Big\{[b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}]u^s_\alpha(\vec{p}) u^{r\dagger}_\beta(\vec{q}) e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + [b^s_{\vec{p}}, c^r_{\vec{q}}]u^s_\alpha(\vec{p}) v^{r\dagger}_\beta(\vec{q}) e^{i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} +$$

$$+ [c^{s\dagger}_{\vec{p}}, b^{r\dagger}_{\vec{q}}]v^s_\alpha(\vec{p}) u^{r\dagger}_\beta(\vec{q}) e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + [c^{s\dagger}_{\vec{p}}, c^r_q]v^s_\alpha(\vec{p}) v^{r\dagger}_\beta(\vec{q}) e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\Big\} =$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\sum_{s,r}[b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}]u^s_\alpha(\vec{p}) u^{r\dagger}_\beta(\vec{q}) e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} - [c^r_q, c^{s\dagger}_{\vec{p}}]v^s_\alpha(\vec{p}) v^{r\dagger}_\beta(\vec{q}) e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} =$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\sum_{s,r}\Big\{(2\pi)^3\delta^{rs}\delta(\vec{p}-\vec{q})\big[u^s_\alpha(\vec{p}) u^{r\dagger}_\beta(\vec{q}) e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + v^s_\alpha(\vec{p}) v^{r\dagger}_\beta(\vec{q}) e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\big]\Big\} =$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sum_s\big[u^s_\alpha(\vec{p}) u^{s\dagger}_\beta(\vec{p}) e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v^s_\alpha(\vec{p}) v^{s\dagger}_\beta(\vec{p}) e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\big] =$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sum_s\big[u^s_\alpha(\vec{p})\bar{u}^s_\lambda(\vec{p}) + v^s_\alpha(-\vec{p})\bar{v}^s_\lambda(-\vec{p})\big](\gamma^0)^\lambda{}_\beta\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} =$$

$$\bar{p}_\mu = (p_0, -\vec{p})$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\big[(\not{p}+m)_{\alpha\lambda} + (\bar{\not{p}}-m)_{\alpha\lambda}\big](\gamma^0)^\lambda{}_\beta\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} =$$

$$\sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\big[b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} - c^s_{\vec{p}} c^{s\dagger}_{\vec{p}}\big]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\big[2p_0\gamma^0 + (p_i\gamma^i - p_i\gamma^i) + (m-m)\big]_{\alpha\lambda}(\gamma^0)^\lambda{}_\beta\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} =$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\,2E_{\vec{p}}\,\delta_{\alpha\beta}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y})$$

Interpretation of commutation relationships and Hamiltonian

In the case of the complex scalar field we attributed the role of annihilation operator to $c_{\vec{p}}$ while the role of creation operator to $c^\dagger_{\vec{p}}$ s.t. $c_{\vec{p}}|0\rangle = 0 \neq c^\dagger_{\vec{p}}|0\rangle$

This was justified by the fact that: $c_{\vec{p}}c^\dagger_{\vec{p}}|0\rangle = c^\dagger_{\vec{p}}c_{\vec{p}}|0\rangle + [c_{\vec{p}}, c^\dagger_{\vec{p}}]|0\rangle = c^\dagger_p c_{\vec{p}}|0\rangle + (2\pi)^3\delta(0)|0\rangle = (2\pi)^3\delta(0)|0\rangle$ i.e. the $c^\dagger_p|0\rangle$ state has a positive norm

That is, in the scalar field case the interpretation of $c^\dagger_{\vec{p}}$ as a creation operator ensures that there exist a positive norm particle state

However, in the case of the spinor field we have: $c_{\vec{p}}c^\dagger_{\vec{p}}|0\rangle = c^\dagger_{\vec{p}}c_{\vec{p}}|0\rangle + [c_{\vec{p}}, c^\dagger_{\vec{p}}]|0\rangle = c^\dagger_{\vec{p}}c_{\vec{p}}|0\rangle - (2\pi)^3\delta(0)|0\rangle$

Therefore, if $c_{\vec{p}}$ is the annihilation operator, particle states have negative norm. We thus have three options:

1) Accept negative norm state
2) Negative norm state is unphysical, c_p is actually the creation operator
3) Reject the harmonic oscillator / canonical commutation relations

To see what to do let's have a look at the Hamiltonian: $H = \sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\big[b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} - c^s_{\vec{p}}c^{s\dagger}_{\vec{p}}\big] = \sum_s\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\big[b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} - c^{s\dagger}_{\vec{p}}c^s_{\vec{p}} + (2\pi)^3\delta(0)\big]$

The commutators are: $[H, b^{s\dagger}_{\vec{p}}] = E_{\vec{p}}b^{s\dagger}_{\vec{p}}$, $[H, b^s_{\vec{p}}] = -E_{\vec{p}}b^s_{\vec{p}}$ and $[H, c^{s\dagger}_{\vec{p}}] = E_{\vec{p}}c^{s\dagger}_{\vec{p}}$, $[H, c^s_{\vec{p}}] = -E_{\vec{p}}c^s_{\vec{p}}$

From the commutators we can see that $c^s_{\vec{p}}$ must be an annihilation operator. If it was a creation operator we would have particles being created by energy decrease and thus the Hamiltonian would not be bounded from below (No Vacuum?)! A negative norm state is also not sensible, it does not lead to a sensible Hilbert space! We thus need to reject this theory and find some new relations!

Fermionic Quantization

The inconsistencies so far encountered are related to the fact that particles described by the Dirac Lagrangian are spin-$1/2$ particles (i.e. Fermions). As we know from Pauli's exclusion principle, fermions wavefunctions are antisymmetric wrt particle exchange as no two identical can occupy the same state.

How does this reflect onto quantization?

When quantizing the real scalar field (i.e. Bosons) no real inconsistencies arose from the use of canonical commutation relations

The Bosonic quantization allowed for $|\vec{p},\vec{q}\rangle = a^\dagger_{\vec{p}} a^\dagger_{\vec{q}}|0\rangle = a^\dagger_{\vec{q}} a^\dagger_{\vec{p}}|0\rangle = |\vec{q},\vec{p}\rangle$ as $[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] \Longrightarrow$ Symmetric w.r.t. exchange of particles

However, we saw that this cannot be the case for fermions. Nonetheless, we can note two things:

1) Pauli's exclusion principle: $|\vec{p},\vec{q}\rangle = -|\vec{q},\vec{p}\rangle$ s.t. $|\vec{p},\vec{q}\rangle + |\vec{q},\vec{p}\rangle = 0$ i.e. $\{c^\dagger_{\vec{p}}, c^\dagger_{\vec{q}}\} = 0$ or $\{b^\dagger_{\vec{p}}, b^\dagger_{\vec{q}}\} = 0$
2) Dirac Lagrangian contains γ matrices which satisfy Clifford Anti-Commutation Algebra

These are signs that Fermions follow anti-commutation

Spin-Statistics Theorem: Bosons (Spin-integer particles) must be quantized according to canonical commutation relations $\Longrightarrow$ Bosonic Quantization

Fermions (Spin-half integer particles) must be quantized according to anticommutation relations $\Longrightarrow$ Fermionic Quantization

Bosons:

$$[a^{a\dagger}_{\vec{p}}, a^{b\dagger}_{\vec{q}}] = [a^a_{\vec{p}}, a^b_{\vec{q}}] = 0 \qquad\qquad [\phi_a(\vec{x},t), \phi_b(\vec{y},t)] = [\Pi_a(\vec{x},t), \Pi_b(\vec{y},t)] = 0$$
$$[a^a_{\vec{p}}, a^{b\dagger}_{\vec{q}}] = \delta^{ab}\,\delta(\vec{p}-\vec{q}) \quad \Longleftrightarrow \quad [\phi_a(\vec{x},t), \Pi_b(\vec{y},t)] = i\delta_{ab}\,\delta(\vec{x}-\vec{y})$$

Fermions:

$$\{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} = (2\pi)^3\delta^{rs}\,\delta(\vec{p}-\vec{q}) \quad \Longleftrightarrow \quad \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = \{\psi^\dagger_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = 0$$
$$\{b^r_{\vec{p}}, b^s_{\vec{q}}\} = \{c^r_{\vec{p}}, c^s_{\vec{q}}\} = \{b^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, c^s_{\vec{q}}\} = \ldots = 0 \qquad\qquad \{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = \delta_{\alpha\beta}\,\delta(\vec{x}-\vec{y})$$

The Dirac Hamiltonian is thus: $H = \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left[b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} + c^{s\dagger}_{\vec{p}} c^s_{\vec{p}} - (2\pi)^3\delta(0) \right]$ Fermion vacuum has negative infinite energy?

We define a vacuum $|0\rangle$ such that $b^s_{\vec{p}}|0\rangle = c^s_{\vec{p}}|0\rangle = 0$

We can then redefine the Hamiltonian w.r.t. the vacuum energy as: $H = \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left[b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} + c^{s\dagger}_{\vec{p}} c^s_{\vec{p}} \right]$

The operators have commutation relations:

$$\left.\begin{aligned} &[H, b^r_{\vec{p}}] = -E_{\vec{p}} b^r_{\vec{p}} \quad \text{and} \quad [H, b^{r\dagger}_{\vec{p}}] = E_{\vec{p}} b^{r\dagger}_{\vec{p}} \\ &[H, c^r_{\vec{p}}] = -E_{\vec{p}} c^r_{\vec{p}} \quad \text{and} \quad [H, c^{r\dagger}_{\vec{p}}] = E_{\vec{p}} c^{r\dagger}_{\vec{p}} \end{aligned}\right\} \begin{aligned} &b^{r\dagger}_{\vec{p}}, c^{r\dagger}_{\vec{p}} \text{ as creation operators} \\ &b^r_{\vec{p}}, c^r_{\vec{p}} \text{ as annihilation operators} \end{aligned}$$

Particle states: $|\vec{p}, r\rangle = b^{r\dagger}_{\vec{p}}|0\rangle \Longrightarrow |\vec{p}_1, r_1; \vec{p}_2, r_2\rangle = b^{r_1\dagger}_{\vec{p}_1} b^{r_2\dagger}_{\vec{p}_2}|0\rangle = -|\vec{p}_2, r_2; \vec{p}_1, r_1\rangle$ Fermi-Dirac Statistics!

Dirac's Interpretation

Dirac's derivation of his famous equation did not arise from group theory and Lagrangians but rather from modifications of Schrödinger's Equation:
Dirac noticed that:

1) Schrödinger's equation is non-relativistic as it is based on the non-relativistic kinetic energy
2) Relativistic theories based on scalar fields do not satisfy total probability conservation

Let's analyze both problems separately

① Schrödinger's equation is non-relativistic

Non-Relativistic free particle (kinetic) energy: $E = p^2/2m \implies$ Non-Relativistic Eq: $i\dot{\psi} = H\psi = (\vec{p}^2/2m)\psi$

Relativistic free particle (kinetic) energy: $E^2 = p^2 + m^2 \implies$ Sch. Equation cannot be relativistic

② Second Order Lagrangians do not conserve probability

Schrödinger's Equation: $i\dot{\psi} = H\psi$

Probability's rate of change: $\dot{P}(t) = \frac{d}{dt}\langle\psi|\psi\rangle = \langle\dot{\psi}|\psi\rangle + \langle\psi|\dot{\psi}\rangle = \langle -iH\psi|\psi\rangle + \langle\psi|-iH\psi\rangle = i\langle\psi|H^\dagger - H|\psi\rangle$

As H is hermitian i.e. $H^\dagger = H$ we have $\dot{P}(t) = 0$ and probability is conserved

Klein-Gordon Equation: $\partial_\mu\partial^\mu \underset{\text{complex field}}{\phi} + m^2\phi = \ddot{\phi} - \vec{\nabla}^2\phi + m^2\phi = 0$

Probability's rate of change: $\dot{P}(t) = \langle\phi|\phi\rangle = \langle\dot{\phi}|\phi\rangle + \langle\phi|\dot{\phi}\rangle \neq 0$ in general

Dirac's Approach

Dirac imposed requirements on the equation

- Must be first order in time
- Hamiltonian must be Hermitian
- Hamiltonian must be able to reproduce $p^2 + m^2$ when squared

He thus modified Sch. equation as follows: $i\dot{\psi} = H\psi = [c\,\vec{\alpha}\cdot\vec{p} + m\beta]\psi$

$$
\begin{aligned}
H^2 = p^2 + m^2 &= c^2(\vec{\alpha}\cdot\vec{p})^2 + m^2\beta^2 + c(\vec{\alpha}\cdot\vec{p})\beta m + m\beta c(\vec{\alpha}\cdot\vec{p}) = \\
&= c^2(p_i\alpha^i)(p_j\alpha^j) + m^2\beta^2 + c\,p_i m\{\alpha^i, \beta\} = \\
&= c^2 p_i p_j \alpha^i\alpha^j + m^2\beta^2 + c\,m\,p_i\{\alpha^i, \beta\}
\end{aligned}
$$

The condition then are: $\alpha^i\alpha^j = \delta^{ij}$ $\quad \beta^2 = 1$ $\quad$ and $\{\alpha^i, \beta\} = 0$

These conditions cannot be satisfied by number but only by matrices: $(\alpha^i, \beta) \implies (-\gamma^0\gamma^i, \gamma^0)$ where $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$ and $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

It follows then that: $i\partial_0\psi = i\gamma^0\gamma^i\partial_i\psi + m\gamma^0\psi \implies (i\gamma^\mu\partial_\mu - m)\psi = (i\not{\partial} - m)\psi = 0$ <u>Dirac Equation!</u>

Interpretation

Dirac derived the equation from the single particle Hamiltonian and thus viewed it as such

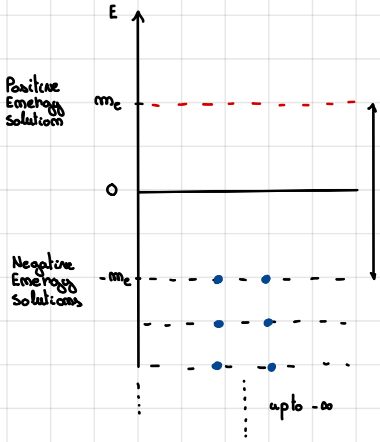
However we know it as a classical field that must be quantized

In the interpretation of ψ as a single particle state, the plane wave solutions are viewed as energy eigenstates

↳ Positive frequency solutions: $\psi = u(\vec{p})e^{-ip\cdot x} \implies i\partial_t\psi = E_{\vec{p}}\psi$ Positive energy solution

Negative frequency solutions: $\psi = v(\vec{p})e^{+ip\cdot x} \implies i\partial_t\psi = -E_{\vec{p}}\psi$ Negative energy solution

The spectrum of solutions is once again unbounded from below as the equation allows for negative energy solutions.
However, as these particles are fermions (by-hand addition by Dirac) they obey Pauli's exclusion principle. Dirac argued that the negative energy states were fully occupied, leaving only positive energy states as observable states and the apparent neutral charge is actually only a relative neutrality wrt the "Dirac Sea" of negative energy states. The fully occupied negative energy states make sense as if there were states available positive energy states would decay to negative energy state. If there were infinite states available decay rate would be infinite (unacceptable)
The Dirac Sea picture made a shocking prediction. When a negative energy state is excited to a positive energy state, a hole is left behind. The hole would have same properties as the electron, positive energy but opposite charge (i.e. positron) as we removed a negative energy state with negative charge



Mass Gap of $2m_e$

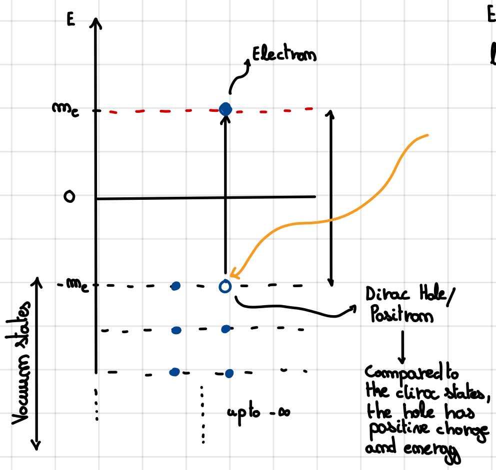
↳ Pair creation can only happen for $E \geq 2m_e$, anything smaller than that cannot lead to pair creation

"Dirac Sea" of completely filled energy states

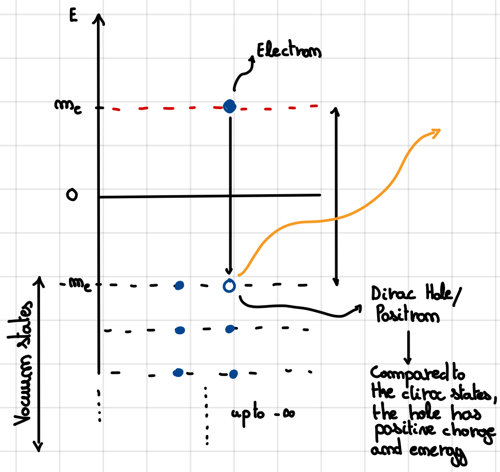
Positive energy states cannot decay to these states because of Pauli's Principle

Pair creation: $\gamma \longrightarrow e^+ + e^-$

Excitation of an electron out of the vacuum leads to the creation of a positron



Pair Annihilation: $e^+ + e^- \longrightarrow \gamma$



"Feynman - Stückleberg Interpretation"

Quantum Field Theory Interpretation

Dirac's interpretation is not completely correct. It is incorrect to view the Dirac equation as a single particle equation. Signs of this can be found in the "Dirac Sea" approach as it sees the existence of anti particles as a purely fermionic characteristics. Fermions and Bosons have both antiparticles. In addition, the Dirac Sea has too many caveats

The correct interpretation views the Dirac equation as the equation of a classical field ψ with positive energy solutions only (H is bounded from below) whose quantization naturally leads to particles and antiparticles being created as result of excitation of the vacuum.

Propagators

Fermionic Propagator $S(x-y)$: $\quad iS(x-y) = \{\psi(x), \bar{\psi}(y)\}$

It follows that: $iS(x-y) = (i\not{\partial}_x + m)\left[D(x-y) - D(y-x)\right]$ where $D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip\cdot(x-y)}$

Causality:

For spacelike intervals i.e. $(x-y)^2<0$ we have $D(x-y) - D(y-x) = 0$

Thus:

Bosons: $[\phi(x), \phi(y)] = 0$ if $(x-y)^2<0 \Longrightarrow$ Operators always commute outside of lightcone

Fermions: $\{\psi_\alpha(x), \psi_\beta(y)\} = 0$ if $(x-y)^2<0 \Longrightarrow$

$\hookrightarrow$ Why?

Away from singularities: $(i\not{\partial}_x - m)S(x-y) = 0$

Computations

$$iS_{\alpha\beta}(x-y) = \{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = \psi_\alpha(x)\psi^\dagger_\lambda(y)(\gamma^0)^\lambda{}_\beta + \psi^\dagger_\lambda(y)(\gamma^0)^\lambda{}_\beta\psi_\alpha(x) =$$

$$= \sum_s\sum_r\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left[\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\}u^s_\alpha(\vec{p})\bar{u}^r_\beta(\vec{q})e^{-i(p\cdot x - q\cdot y)} + \{b^s_{\vec{p}}, c^r_{\vec{q}}\}u^s_\alpha(\vec{p})\bar{v}^r_\beta(\vec{q})e^{-i(p\cdot x + q\cdot y)}\right] +$$

$$+ \sum_s\sum_r\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left[\{c^{s\dagger}_{\vec{p}}, b^{r\dagger}_{\vec{q}}\}v^s_\alpha(\vec{p})\bar{u}^r_\beta(\vec{q})e^{i(p\cdot x + q\cdot y)} + \{c^{s\dagger}_{\vec{p}}, c^r_{\vec{q}}\}v^s_\alpha(\vec{p})\bar{v}^r_\beta(\vec{q})e^{i(p\cdot x - q\cdot y)}\right] =$$

$$= \sum_s\sum_r\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\left[(2\pi)^3\delta^{rs}\delta(\vec{p}-\vec{q})\left(u^s_\alpha(\vec{p})\bar{u}^r_\beta(\vec{q})e^{-i(p\cdot x - q\cdot y)} + v^s_\alpha(\vec{p})\bar{v}^r_\beta(\vec{q})e^{i(p\cdot x - q\cdot y)}\right)\right] =$$

$$= \sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[(\not{p}+m)_{\alpha\beta}e^{-ip\cdot(x-y)} + (\not{p}-m)_{\alpha\beta}e^{-ip(y-x)}\right] =$$

$$= \sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[(\gamma^\mu p_\mu)_{\alpha\beta}\left(e^{-ip(x-y)} + e^{-ip(y-x)}\right) + m\left(e^{-ip\cdot(x-y)} - e^{-ip(y-x)}\right)\right] =$$

$$= \left[i(\gamma^\mu)_{\alpha\beta}\partial_\mu + m\right]\left[D(x-y) - D(y-x)\right] = \left[i(\not{\partial})_{\alpha\beta} + m\right]\left[D(x-y) - D(y-x)\right]$$

$$(i\not{\partial}_x - m)S(x-y) = -i(i\not{\partial}_x - m)iS(x-y) = -i(i\not{\partial}_x - m)(i\not{\partial}_x + m)\left[D(x-y) - D(y-x)\right] =$$

$$= i(\not{\partial}^2_x + m^2)\left[D(x-y) - D(y-x)\right] = i\left[(\gamma^\mu\partial_{\mu,x})^2 + m^2\right]\left[D(x-y) - D(y-x)\right] =$$

$$= i\sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left\{\left[(\gamma^\mu)^2(-i)^2(p_\mu p^\mu) + m^2\right]e^{-ip(x-y)} + \left[(\gamma^\mu)^2(i)^2(p_\mu p^\mu) + m^2\right]e^{-ip(y-x)}\right\} =$$

$$= i\sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left\{\left[-(\gamma^0)^2p_0^2 + (\gamma^i)^2p_i^2 + m^2\right]e^{-ip(x-y)} + \left[-(\gamma^0)^2p_0^2 + (\gamma^i)^2p_i^2 + m^2\right]e^{-ip(y-x)}\right\} =$$

$$= i\sum_s\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[E^2_{\vec{p}} - p_0^2\right]\left(e^{-ip(x-y)} + e^{-ip(y-x)}\right) = 0 \text{ for on shell calculations}$$

Dirac Eq

$$(i\not{\partial}_x - m)S(x-y) = -i(i\partial_x - m)\{\psi(x), \bar{\psi}(y)\} = -i(i\not{\partial}_x - m)\left(\psi(x)\bar{\psi}(y) + \bar{\psi}(y)\psi(x)\right) = -i\left[(i\not{\partial}_x - m)\psi(x)\right]\bar{\psi}(y) - i\bar{\psi}(y)\left[(i\not{\partial}_x - m)\psi(x)\right] = 0$$